UNIVERSIDADE TÉCNICA DE LISBOA

INSTITUTO SUPERIOR DE ECONOMIA E GESTÃO

Mestrado em: Matemática Financeira

COPULAS AND DEFAULTS WITHIN A CRISIS

CLÁUDIA CATARINA ACÚRCIO DUARTE

Orientação: Doutora Ana Isabel Pires Sarmento Lacerda

Júri:

Presidente: Doutor Onofre Alves Simões

Vogais: Doutor João Carlos Henrique da Costa Nicolau

Doutora Ana Isabel Pires Sarmento Lacerda

Maio/2010

Abstract

In the aftermath of the subprime crisis, the main purpose of this thesis is to assess the default dependency among firms, studying the case of four US financial institutions in two periods of time: before and during the crisis. The methodology followed is based on conditional copula models, which provides a set of global and tail dependency measures, beyond the linear correlation widely misused in financial problems. For this purpose, we use CDS (credit default swap) data to estimate the copulas, that are assumed to be a proxy for default closeness, as they reflect the credit risk of the institutions. As far as we know, this is a novelty of the present analysis. The usual practice is to use equity returns, which are incomplete and more indirect indicators of defaults. The procedures are carried out in two steps. First, we model the individual dynamics for defaults closeness, by using ARMA-GARCH specifications applied to CDS spreads variations and assuming t-distributed innovations (to capture the extreme observations). Then, we fit a set of copula functions to the standardised residuals of the marginal distributions. The best specifications for the characterisation of the dependency structure are different for the two subperiods analysed, confirming a structural break in the default dependency pattern, occurred in the summer of 2007. The results also confirm our expectations regarding the global dependency under stressful scenarios. For the four considered financial institutions, all the dependency measures rose substantially in the crisis period. Furthermore, we observe a significant increase in the upper tail dependency, corresponding to high probabilities of simultaneous defaults. These outcomes point out to the increase of the systemic and contagion risks in the US financial markets.

Keywords: Crisis, Copula, Default, Dependency, CDS spreads, ARMA-GARCH.
JEL Classification: C22, C46, G01, G21.

Resumo

No rescaldo da crise do *subprime*, o principal objectivo desta tese é avaliar a dependência entre os *defaults* de empresas, estudando o caso de quatro instituições financeiras americanas em dois períodos: antes e durante a crise. A metodologia utilizada baseia-se em modelos de cópulas condicionadas, que fornecem um conjunto de medidas de dependência global e de cauda, complementando a correlação linear indevidamente utilizada nos problemas financeiros. Para esse efeito, usamos os CDS para estimar cópulas, que são assumidos como *proxy* para a proximidade ao *default*, dado que reflectem o risco de crédito das instituições. Tanto quanto sabemos, esta é uma novidade da presente análise. A prática usual é utilizar rendibilidades das acções, que são um indicador incompleto e mais indirecto dos *defaults*. Os procedimentos são realizados em duas etapas. Primeiro, são modelizadas as dinâmicas individuais para a proximidade aos *defaults*, usando especificações ARMA-GARCH para as variações dos CDS spreads e assumindo que as inovações seguem a distribuição t-student (para captar as observações extremas). De seguida, ajustamos um conjunto de funções cópula para os resíduos standardizados das distribuições marginais. As melhores especificações para a caracterização da estrutura de dependência são diferentes para os dois sub-períodos analisados, confirmando uma quebra estrutural no padrão de dependência dos *defaults*, ocorrida no Verão de 2007. Os resultados também confirmam as nossas expectativas relativamente à dependência global em cenários de stress. Para os quatro bancos considerados, todas as medidas de dependência aumentaram substancialmente no período de crise. Além disso, observamos um aumento significativo da dependência na cauda direita, o que corresponde a uma elevada probabilidade de *defaults* simultâneos. Estes resultados apontam para o aumento do risco sistémico e de contágio nos mercados financeiros dos EUA.

Palavras-Chave: Crise, Copula, Default, Dependência, CDS spreads, ARMA-GARCH.
Classificação JEL: C22, C46, G01, G21.

Contents

List of Tables

List of Figures

Nomenclature

ACF	Autocorrelation function
ADF	Augmented Dickey-Fuller
AIC	Akaike's information criteria
ARCH	Autoregressive conditional heteroscedasticity
ARMA	Autoregressive moving average
BIS	Bank for International Settlements
bp	basis points
CDS	Credit default swap
CML	Canonical maximum likelihood
DF	Dickey-Fuller
GARCH	Generalised autoregressive conditional heteroscedasticity
IFM	Inference functions for margins
IGARCH	Integrated generalised autoregressive conditional heteroscedasticity
iid	Independent and identically distributed
LB	Ljung-Box
ML	Maximum likelihood
PACF	Partial autocorrelation function

Acknowledgments

Agradecimentos

- ⋆ À minha orientadora, Ana Lacerda, pela dedicação pessoal, disponibilidade e empenho na supervisão da tese.
- ⋆ Aos colegas da *Área de Análise e Controlo Financeiro da Gestão de Activas e Reservas* do Banco de Portugal, pelo apoio incondicional, compreensão e tempo disponibilizado. Em particular, ao colega João Pedro Brito, pela revisão da tese e sugestões pertinentes.
- ⋆ Ao Bruno Albuquerque, quer pela amizade demonstrada, quer pela leitura do texto e respectivos comentários, sem esquecer os bons momentos proporcionados pelas discussões :-)
- ⋆ Ao António Antunes, sempre prestável e ajuda imprescindível em tantas dúvidas que surgiram.
- ⋆ À Dora Leal, pelas trocas de impressões muito úteis sobre a metodologia.
- ⋆ To Rafael Guitart, by his support and useful comments regarding the writing procedures.
- ⋆ Ao colega de mestrado André Ribeiro, por ouvir os desabafos em tantos momentos de desânimo.
- ⋆ Aos amigos em geral, pela atenção e paciência nesta fase, especialmente à Ângela Salvador, que lidou diariamente com o problema :-)
- ⋆ Aos meus pais, que me apoiaram em todas as dificuldades e foram os grandes responsáveis pela manutenção da minha sanidade mental, ao longo do percurso efectuado!
- ⋆ À Fundação Económicas, pela comparticipação financeira do mestrado.
- ⋆ A todos os que contribuiram directa ou indirectamente para o trabalho, que não menciono aqui para não me arriscar a escrever mais páginas de agradecimentos do que de trabalho!

Chapter 1

Introduction

How could this happen? No one thought that the financial system could collapse... The modern financial system is immensely complex - possible too complex for any person to really understand it. Interconnections create systemic risks that are extraordinary difficult to figure out. The fact that things apparently worked so well ... gave everyone a false sense of comfort.[1]

Economic crises are the result of years, even decades, of global economic change, imbalances accumulation, policy errors and investor misjudgement. Although these crisis never happen overnight, there is always a moment when the world seems to be upside down, faced with a dramatic loss of confidence. The bankruptcy of the 158-years-old firm Lehman Brothers, on 15 September 2008, was indeed one of those such moments, corresponding to the worst point of the financial crisis and plunging the world economy in to what seems to be the most dangerous recession, since the Great Depression of the late 1920s.

Modern life style requires an appropriate and reliable functioning of the financial system, composed of banks, insurance companies, governments, pension funds and securities firms. Around the middle of the current decade, the financial world system seemed to be working perfectly. Prosperity and stabilisation were evidenced in the main economic indicators, with inflation kept at low levels and economic growth ex-

[1] BIS (2009), page 4.

hibiting a sustainable rate. In addition, central banks were providing liquidity to the market, lending to banks when needed, deposit insurance and investors protections schemes were in place and regulators and supervisors were, apparently, comfortable with the risk taken by financial institutions.

However, when the house prices began their steep decline, after their peak in mid-2006, refinancing became more difficult. Securities backed with subprime mortgages,[2] widely held by financial institutions, lost most of their value resulting in a large decline in the capital of many banks and U.S. government sponsored agencies, in particular, Freddie Mac and Fannie Mae. The subprime crisis had begun. The losses were even higher than expected, as many banks have hidden their holdings of sub-prime mortgages in exotic, off-balance sheet instruments such as structured investment vehicles. The lack of confidence between banks, evidenced by the unwillingness to lend to each other, was a clear sign of the collapse, prevailing a huge panic in financial markets and in economies in general. Bear Stearns was rescued in March 2008[3] and, six months later, it was the Lehman Brothers bankruptcy. In fact, since then, the complex financial system has been critically hampered to its core. Public bailouts of financial institutions and massive economic support packages as well as unconventional liquidity support measures by central banks, to restore the global financial markets, remind the vulnerability of the system and the importance of keeping it healthy. This unprecedent intervention of central banks and governments was fundamental to avoid a more severe and lasting recession. However, such monetary policies and government actions, for instance, the very low level of interest rates, bank rescues, fiscal stimulus programs and bailouts, cannot be maintained for too long, to avoid future asset-price bubbles and other destabilising phenomena.

Media, literature, investigators often inquire on the causes of the current crisis. The increased risk-taking and leverage, due to the long period of low real interest rates are commonly presented as reasons. The global imbalances and the dependency

[2]The subprime home loans are provided to borrowers with poor credit histories and weak documentation of income.

[3]The Federal Reserve (FED) engineered a bailout which led to the acquisition of Bear Stearns by JP Morgan, in which the FED insured a significant amount of the investment bank's structured finance portfolio.

between the export emerging countries and industrial economies that, at least, worsened and extended the crisis reach and duration. From a microeconomic point of view, several factors were also pointed out. Among them, it should be stressed, on the one hand, the dangerous incentives for investors and for financial sector employees and, on the other hand, the lack of impartiality in ratings assignment, depending on who pays and needs them the most; the flaws in techniques[4] used to measure, price and manage risk and in corporate governance structures used to monitor it; and the shortfalls of the regulatory systems.

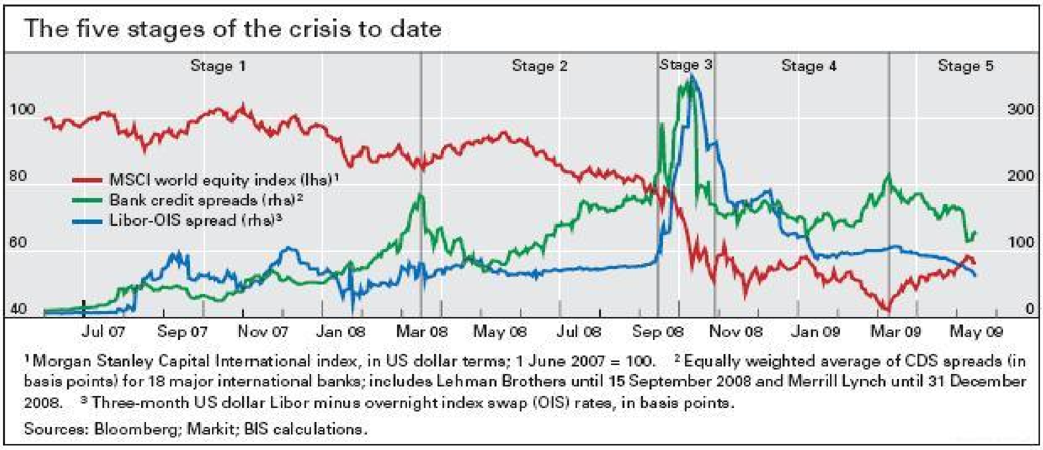


Figure 1.1: Five stages of the crisis, BIS (2009) page 16.

The aforementioned reasons were the ground for a severe financial crisis, with deep implications on the economic activity. Figure 1 presents an indicative sectioning of the crisis period:

1. Pre-March 2008: prelude of the crisis, leading up to the takeover of Bear Sterns;

2. Mid-March to mid-September 2008: towards the Lehman bankruptcy, credit default swaps (CDS) spreads increased substantially;

3. 15 September to late October 2008: global loss of confidence, a growing number of financial institutions were facing risk of default, large scale bank rescues, deposit and debt guarantees;

[4] Mathematics and the mathematical models used by the finance industry have been criticised in some journalistic articles. See Donnelly and Embrechts (2010) for an answer on these criticisms.

4. Late October 2008 to mid-March 2009: global downturn - declines in GDP, inflation falls, huge fiscal stimulus packages were announced by governments, rates cut to near zero levels, outright purchases of public and corporate debt;

5. Since mid-March 2009: downturn deepens but loses speed, first signs of stabilisation, volatilities have declined but persistent signs of dysfunction in markets remain.

In this context, for the years 2007-2008, the following key events should be highlighted:

⋆ July/August 2007 - Liquidity crisis

⋆ November 2007 - Northern Rock collapse

⋆ March 2008 - Bear Sterns takeover

⋆ September/October 2008 - Lehman bankruptcy

The behaviour of the average CDS spreads for 18 major international banks (see figure 1) reflect these key events, seeming to provide a reliable indicator of the magnitude of the crisis.

Under this scenario, an accurate measure of credit risk and default dependency between institutions became an absolute need. Lehman Brothers filled the largest bankruptcy in history and it was followed by a large number of institutions. This phenomenon was not predicted in credit risk products whose pool of collateral eventually did not include Lehman, but included some of related institutions' debt that certainly were affected by the Lehman fall-out. Hence, estimating dependencies is really important, because under stressful conditions the mispecification of diversification can be crucially dangerous to portfolio returns. Throughout the crisis, the early warning signals of deteriorating credit quality, materialised in CDS spreads, as well as the evidence of bankruptcies are obvious issues to study.

In this context, the main goal of the thesis is to understand the association and default dependencies among financial institutions, before and during the crisis, testing a structural break between these two periods. We use CDS data in dependency modelling, which is a novelty of the present work, as far as we know. An usual practice is to assume that default probabilities share the same dependency structure than equity returns and use them to fit the copula.[5] However, in this thesis, CDS spreads are used because they are a more direct and complete indicator of defaults than equity data, confirmed by their increasing liquidity of the last years. In fact, when the price of a CDS goes up, it indicates that default risk has risen. Our analysis is carried out using the CDS spreads of four US institutions, namely, JP Morgan Chase & Co., Goldman Sachs Inc, Bank of America Corp and Citigroup Inc. The relationship between these institutions is studied for two periods: before the crisis, since January 2006 until July 2007, and during the crisis, since August 2007 until March 2009. The work follows Dias and Embrechts (2003) and Palaro and Hotta (2006) to explore dependency concepts through conditional copula models. Firstly, the individuals default behaviour is specified using the CDS spreads variations of the corresponding institutions to fit ARMA-GARCH models. Then, based on conditional copula-approach, the residuals of default dynamics are the inputs used to evaluate the default dependency between the considered institutions. The measures of dependency based on rank correlation and tail dependency are taken into account beyond the correlation estimates commonly used.

The remainder of the text is organised as follows. Chapter 2 describes the theoretical framework needed to understand copulas as a tool to capture dependencies between random variables. In particular, in section 2.1 is presented a survey on copula related definitions, properties and dependency measures, such as global and tail dependency, and their suitability compared to linear correlation. In addition, we review a set of elliptical and Archimedean copula specifications, as well as the estimation procedures based on maximum likelihood. At the end of the section, the conditional copula framework is introduced. In section 2.2, the application of copulas to finance and some references are pointed out. Chapter 3 presents all the empirical study, repeating the procedures for the two periods considered, before and during the crisis, for each of the six pairs composed by four US financial institutions:

[5] See Elizalde (2005) for more details.

JP Morgan Chase, Goldman Sachs, Bank of America and Citigroup. In section 3.1, the CDS data is analysed and all the pre-model work is explained, including the differentiation of CDS spreads to obtain stationary series. Section 3.2 implements the individual dynamics for CDS spreads variations, through ARMA-GARCH models, assuming t-distribution for innovations, and tests their adequacy. In addition, a set of copula models are fitted evaluating the t-cumulative distribution function at the standardised residuals from the ARMA-GARCH modelling. The AIC criterion ranks the estimated copulas and the confidence intervals are calculated to ensure robustness of copula parameters estimates. Given the selected copulas, we interpret the global and tail dependency measures before and during the crisis, for the pairs of institutions considered. Finally, section 4 presents a summary of the results and suggestions of future investigation. In appendix A we illustrate a particular dynamic copula which is compared with the corresponding static one, to motivate the study of other specifications and possibilities.

Chapter 2

Theoretical framework

This chapter provides an introduction to the copula theory. In section 2.1 we review the theoretical background on copula function, while in section 2.2 some references of empirical applications to finance are presented.

2.1 Copulas

The term copula comes from the Latin noun which means "link, tie, bond", *i.e.*, joining together. A copula is a function which *marries* a group of univariate marginal distribution functions into a multivariate distribution function, capturing the relationship between random variables. Every joint distribution function of a set of risk factors implicitly contains a description of the marginal behaviour of individual risk factors and a description of their dependency structure. Copula functions provide a way of isolating the dependency structure and to express it on a *quantile scale*, which is useful for describing the dependency of extreme outcomes and, naturally, to use in a risk-management context.

Copula functions, introduced in 1959, are the most general way to view dependency of random variables. They provide a number of useful alternative measures of dependency to the linear correlation coefficient, which often fail to capture important risks. In fact, correlation is widely misused in finance, being applied to problems for which is not suitable. The linear (Pearson) correlation coefficient, which belongs to the interval $[-1, 1]$, is defined as

$$\rho(Y_1, Y_2) = \frac{Cov(Y_1, Y_2)}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}},$$

where Y_1 and Y_2 are random variables.

The linear correlation coefficient merely captures the linear dependency. Two perfectly dependent variables exhibit a correlation coefficient of $+1$ or -1, depending on the variables being positively (*i.e.*, rise or fall together) or negatively dependent, respectively. The closer the coefficient is to either -1 or 1, the stronger the correlation. If the variables are independent, the linear correlation coefficient is zero, but the converse is not true, with the exception being the variables whose joint distribution is normal. For other cases the linear correlation can be an imperfect measure of dependency. This measure presents several limitations, as it is not invariant under non-linear transformations of the random variables and it is not defined when variances approach to infinite, which is the case of heavy-tailed distributions. As in financial series there is strong dependency among big losses (gains), this is an important drawback of correlation.

Furthermore, according to Elizalde (2005), the conclusions extracted from the comparison of linear default correlations should be read carefully. In fact, these measures are covariance-based and do not make sense for joint non-elliptical random variables[1] such as default events. The use of copula based measures of dependency can overcome this limitation. Unlike correlation, copulas are invariant under strictly increasing transformations of risks and allows us to model asymmetries, as we will see below. For a more detailed description of copula theory see McNeil et al. (2005), Nelsen (1999) and Embrechts et al. (2001).

[1]If Y is a n-dimensional random vector, for some $\mu \in \mathbb{R}^n$ and some $n \times n$ nonnegative definite, symmetric matrix Σ, the characteristic function $\psi_{Y-\mu}(t)$ of $(Y-\mu)$ is a function of the quadratic form $t^T \Sigma t$. So Y has an elliptical distribution with parameters μ, Σ and ψ. t-student and normal distributions are examples of elliptical distributions. Even for jointly elliptically distributed random variables there are situations where using linear correlation does not make sense, for example, when there is infinite second moments such as in a t_2 distribution.

2.1.1 Definition

A copula function $\mathbf{C}$ is the joint distribution of a set of N uniform random variables $U_1, \ldots, U_N$, allowing to separate the modelling of the marginal distribution functions from the modelling of the dependency structure. The choice of the copula does not constrain the choice of the marginal densities and *vice-versa*. Moreover, copulas differ not so much in the degree of association they provide, but rather in which part of the distributions the association is strongest.

A formal definition of a copula function is as follows:

Definition 1. *A N-dimensional copula is a function $\mathbf{C} : [0,1]^N \rightarrow [0,1]$ satisfying the conditions:*

- *For all $(u_1, \ldots, u_N)$ in $[0,1]^N$, if at least one component u_i is zero, then $\mathbf{C}(u_1, u_2, \ldots, u_N) = 0$.*
- *For $u_i \in [0,1]$, $C(1, \ldots, 1, u_i, 1 \ldots, 1) = u_i$ for all $i \in \{1, 2, \ldots, N\}$.*
- *For all $[u_{11}, u_{12}] \times [u_{21}, u_{22}] \times \ldots \times [u_{N1}, u_{N2}]$ N-dimensional rectangles in $[0,1]^N$,*

$$\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_N=1}^{2} (-1)^{i_1+i_2+\ldots+i_N} \mathbf{C}(u_{1i_1}, u_{2i_2} \ldots, u_{Ni_N}) \geq 0.$$

An alternative and more intuitive definition of a copula function is presented in Schönbucher (2003):

Definition 2. *A function $\mathbf{C} : [0,1]^N \rightarrow [0,1]$ is a copula if there are uniform random variables $U_1, \ldots, U_N$ taking values in $[0,1]$ such that $\mathbf{C}$ is their joint distribution function.*

The Sklar's Theorem(1959) shows that any multivariate distribution function F can be written as a copula function. Formally,

Theorem 3. ***(Sklar)*** *Let $Y_1, Y_2, \ldots, Y_N$ be random variables with marginal distribution functions $F_1, F_2, \ldots, F_N$ and joint distribution function F. Then, there is a*

N-dimensional copula $\mathbf{C}$ *such that for* $(y_1, y_2, \ldots, y_N) \in \mathbf{R^N}$ *we have*

$$F(y_1, y_2, \ldots, y_N) = \mathbf{P}[Y_1 \leq y_1, Y_2 \leq y_2, \ldots, Y_N \leq y_N] = \mathbf{C}(F_1(y_1), F_2(y_2), \ldots, F_N(y_N)). \tag{2.1}$$

Additionally, if each F_i *is continuous, the copula* $\mathbf{C}$ *is unique.*

Sklar's theorem expresses the basic idea of dependency modelling via copula functions by stating that, for any multivariate distribution function, the univariate margins (the distribution functions of random variables) and the dependency structure can be separated, with the latter being completely described by a copula function. This theorem has the following important corollary for simulation purposes:

Corollary 4. *Let* F *and* $\mathbf{C}$ *be, respectively, a N-dimensional distribution function (with continuous univariate margins* $F_1, F_2, \ldots, F_N$*) and a N-dimensional copula function. Then, for any* $u \in [0,1]^N$*, we have*

$$C(u_1, u_2, \ldots, u_N) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_N^{-1}(u_N)), \tag{2.2}$$

where $F_i^{-1}(u_i)$ *denotes the inverse of the cumulative distribution function, namely, for* $u_i \in [0,1], F_i^{-1}(u_i) = \inf\{y : F_i(y) \geq u_i\}$.

Corollary 5. *Applying Sklar's theorem and using the relation between the distribution and the density function, we can derive the multivariate copula density* $c(F_1(y_1), \ldots, F_N(y_N))$ *associated with a copula function* $\mathbf{C}(F_1(y_1), \ldots, F_N(y_N))$*:*

$$\begin{aligned} f(y_1, \ldots, y_N) &= \frac{\partial^N [\mathbf{C}(F_1(y_1), \ldots, F_N(y_N))]}{\partial F_1(y_1) \ldots \partial F_N(y_N)} \cdot f_1(y_1) \ldots f_N(y_N) \\ &= c(F_1(y_1), \ldots, F_N(y_N)) \cdot f_1(y_1) \ldots f_N(y_N), \end{aligned} \tag{2.3}$$

where we define

$$c(F_1(y_1), \ldots, F_N(y_N)) = \frac{\partial^N [\mathbf{C}(u_1, \ldots, u_N)]}{\partial u_1 \ldots \partial u_N} = \frac{f(y_1, \ldots, y_N)}{f_1(y_1) \ldots f_N(y_N)}.$$

The associated copula density is particulary useful to calibrate its parameters to real market data.

Remark 6. *Consider a pair of uniform* $(0,1)$ *random variables* (U,V) *with copula* $\mathbf{C}$. *We have*

$$P(V \leq v|U=u) = \frac{\partial}{\partial u}\mathbf{C}(u,v); \tag{2.4}$$

$$\mathbf{C}(u,1) = \mathbf{P}(U \leq u, V \leq 1) = \mathbf{P}(U \leq u) = u; \tag{2.5}$$

and

$$\mathbf{C}(u,0) = \mathbf{P}(U \leq u, V \leq 0) = 0. \tag{2.6}$$

2.1.2 Measures of dependency

Since each marginal distribution F_i contains all the univariate information on the individual variable Y_i and the joint distribution function F contains all the univariate and multivariate information, the information contained on the copula $\mathbf{C}$ respects the dependency between the marginal distributions (Y_i variables).

The dependency between variables can be assessed with different measures. On the one hand, we have the linear correlation and, on the other hand, we have copula based dependency measures, such as rank correlation coefficients (Kendall's tau and Spearman's rho) and coefficients of tail dependency. Some measures of association are only dependent on the copula and not on the marginal distributions. As we have already mentioned, the linear correlation coefficient, the most used measure of dependency, measures the overall strength of the association but does not provide information on the variation of this association across the distribution. Furthermore, unlike linear correlation, rank correlation and tail dependency coefficients do not depend on the marginal distributions.

The characteristics of the data and the respective dependency measures can suggest the copula specification to choose. In particular, these measures indicate the part of the distributions where variables are more associated, specially in the tails, which represent, for example, correlation among large losses.

As in the application presented in this work only considers pairs of variables, a significant part of the theoretical framework, including the dependency measures, is explained for the bivariate case.

Rank correlation

Rank correlations are simple scalar association measures that only depend on the copula of a bivariate distribution. The standard empirical estimators of rank correlation may be calculated only by analysing the ranks of the data, regardless of the actual numerical values.

Definition 7. ***Concordance*** *Let* (y_1, y_2) *and* (y_1', y_2') *be two observations from a vector* (Y_1, Y_2) *of continuous random variables. Then,* (y_1, y_2) *and* (y_1', y_2') *are said to be concordant if* $(y_1 - y_1')(y_2 - y_2') > 0$ *and discordant if* $(y_1 - y_1')(y_2 - y_2') < 0$*.*

Intuitively, if Y_2 tends to increase with Y_1, then we expect that the probability of concordance to be high relative to the probability of discordance; if Y_2 tends to increase with the decreasing of Y_1, then we expect the opposite.

Kendall's tau and Spearman's rho are two measures of dependency, based on the concordance concept.

Definition 8. *Let* (Y_1, Y_2) *and* (Y_1', Y_2') *be i.i.d. random vectors of continuous random variables with the same joint distribution function given by the copula* $\mathbf{C}$ *(and with marginals* F_1 *and* F_2*). Then, Kendall's tau of the vector* (Y_1, Y_2) *(and of the copula* $\mathbf{C}$*) is defined as the probability of concordance minus the probability of discordance,* i.e.

$$
\begin{aligned}
\tau &= \mathbf{P}[(Y_1 - Y_1')(Y_2 - Y_2') > 0] - \mathbf{P}[(Y_1 - Y_1')(Y_2 - Y_2') < 0] \qquad (2.7)\\
&= 4\int\int_{[0,1]^2} \mathbf{C}(u,v)d\mathbf{C}(u,v) - 1\\
&= 1 - 4\int\int_{[0,1]^2} \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v}dudv. \qquad (2.8)
\end{aligned}
$$

Definition 9. *Let* (Y_1, Y_2)*,* (Y_1', Y_2') *and* (Y_1'', Y_2'') *be i.i.d. random vectors of continuous random variables with the same joint distribution function given by the copula* $\mathbf{C}$ *(and with marginals* F_1 *and* F_2*). Then, Spearman's rho of the vector* (Y_1, Y_2) *(and* *of the copula* $\mathbf{C}$*) is defined as*

$$\begin{aligned}\rho_S &= 3(\mathbf{P}[(Y_1 - Y_1')(Y_2 - Y_2'') > 0] - \mathbf{P}[(Y_1 - Y_1')(Y_2 - Y_2'') < 0]) && (2.9)\\ &= 12\int\int_{[0,1]^2} uvd\mathbf{C}(u,v) - 3\\ &= 12\int\int_{[0,1]^2} \mathbf{C}(u,v)dudv - 3. && (2.10)\end{aligned}$$

Spearman's rho can be interpreted as the linear correlation between distribution functions of random variables. Let (Y_1, Y_2) be a random vector of continuous random variables with the same joint distribution function H (whose margins are F_1 and F_2) and copula $\mathbf{C}$, and consider the random variables $U = F(Y_1)$ and $V = F(Y_2)$. Therefore, we can write the Spearman's rho coefficient of (Y_1, Y_2) as

$$\begin{aligned}\rho_S(Y_1, Y_2) &= 12\int\int_{[0,1]^2} uvd\mathbf{C}(u,v) - 3 = 12\mathbf{E}[UV] - 3 = \frac{\mathbf{E}[UV] - 1/4}{1/12}\\ &= \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \rho(U,V) = \rho(F_1(Y_1), F_2(Y_2)),\end{aligned}$$

where ρ denotes the linear correlation coefficient. So, the Spearman's rho of the vector (Y_1, Y_2) is the Pearson correlation of the random variables $F_1(Y_1)$ and $F_2(Y_2)$.

These rank measures of dependency take values that belong to the interval $[-1, 1]$. It equals -1 if the two random variables are countermonotonic; equals 1 if they are comonotonic and, equals 0 if they are independent.

Tail dependency

Kendall's tau and Spearman's rho are measures of global dependency. In contrast, tail dependency coefficients between two random variables (Y_1, Y_2) are local/extremal measures of dependency, as they refer to the level of dependency between extreme values, *i.e.*, the tails of the distributions $F_1(Y_1)$ and $F_2(Y_2)$.

Copulas have a flexible structure that allow for tail dependency, which is a very important feature to study correlated defaults in crisis periods. The concept of tail dependency is specified for each tail and it relates to the amount of dependency in the upper right quadrant tail or/and lower left quadrant tail of a bivariate distribution, addressing upper tail dependency, lower tail dependency or both.

The coefficients we describe are defined in terms of limiting conditional probabilities of *quantile exceedances*.

Definition 10. *Let* (Y_1, Y_2) *be a random vector of continuous random variables with copula* $\mathbf{C}$ *(and with marginals* F_1 *and* F_2*). Then, the coefficient of upper tail dependency of the vector* (Y_1, Y_2) *(and of the copula* $\mathbf{C}$*) is defined as*

$$\begin{aligned}
\lambda_U &= \lim_{u \to 1} \mathbf{P}[Y_1 > F_1^{-1}(u) | Y_2 > F_2^{-1}(u)] \qquad (2.11)\\
&= \lim_{u \to 1} \frac{1 - \mathbf{P}[Y_1 \leq F_1^{-1}(u)] - \mathbf{P}[Y_2 \leq F_2^{-1}(u)] + \mathbf{P}[Y_1 \leq F_1^{-1}(u), Y_2 \leq F_2^{-1}(u)]}{1 - \mathbf{P}[Y_2 \leq F_2^{-1}(u)]}\\
&= \lim_{u \to 1} \frac{1 - 2u + \mathbf{C}(u,u)}{1-u} \qquad (2.12)
\end{aligned}$$

where F_i^{-1} *represents the inverse function of* F_i*, provided the limit exist. We say that the random vector (and copula* $\mathbf{C}$*) has upper tail dependency if* $\lambda_U > 0$*. Similarly, the coefficient of lower tail dependency of the vector* (Y_1, Y_2) *(and copula* $\mathbf{C}$*) is defined as*

$$\lambda_L = \lim_{u \to 0} \mathbf{P}[Y_1 < F_1^{-1}(u) | Y_2 < F_2^{-1}(u)] = \lim_{u \to 0} \frac{\mathbf{C}(u,u)}{u}. \qquad (2.13)$$

We say that the random vector (and copula $\mathbf{C}$*) has lower tail dependency if* $\lambda_L > 0$*.*

Upper (lower) tail dependency measures the probability of a component of the vector (Y_1, Y_2) to be extremely large (small) given that the other is extremely large (small). Intuitively, upper (lower) tail dependency exists when there is a positive probability of positive (negative) outliers occurring jointly.

If $\lambda_U = 0$ ($\lambda_L = 0$), then the two random variables (Y_1, Y_2) are said to be asymptotically independent in the upper (lower) tail.

2.1.3 Copula Families

This subsection surveys some of the copula functions most used in default risk modelling. First, we present the Gaussian and T-copulas, which belong to the elliptical family of copulas. Then, we describe the class of Archimedean copulas.

Multivariate Gaussian Copula

The N-dimensional Gaussian copula with covariance matrix Σ is given by

$$\mathbf{C}(u_1, \ldots, u_N) = \Phi_\Sigma^N(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N)), \tag{2.14}$$

where Φ_Σ^N represents a N-dimensional normal distribution function, with Σ and Φ^{-1} denoting the covariance matrix and the inverse of the univariate normal distribution function, respectively.

Normal copulas are radially symmetric ($\lambda_U = \lambda_L$), tail independent ($\lambda_U = \lambda_L = 0$). Furthermore, the linear correlation coefficient ρ can be expressed in terms of both Kendall's tau (τ) and Spearman's rho (ρ_S): $\rho = 2\sin(\frac{\pi}{6}\rho_S) = \sin(\frac{\pi}{2}\tau)$.

As with any other copula, the normal copula enables the use of any marginal distribution.

Multivariate Student's T-Copula

Let X be a random vector distributed as a N-dimensional multivariate t-student with ν degrees of freedom, mean vector μ (for $\nu > 1$) and covariance matrix $\frac{\nu}{\nu-2}\Sigma$ (for $\nu > 2$). X can be expressed as

$$X = \mu + \frac{\sqrt{\nu}}{\sqrt{S}} Z, \tag{2.15}$$

where S is a random variable χ^2-distributed with ν degrees of freedom and Z is a N-dimensional normal random vector, independent of S, with zero mean and covariance matrix Σ. The N-dimensional T-copula of X is given by

$$\mathbf{C}(u_1, \ldots, u_N) = t_{\nu,R}^N(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_N)), \tag{2.16}$$

where $t_{\nu,R}^N$ is the distribution function of $\frac{\sqrt{\nu}}{\sqrt{S}}Z$, Z is a N-dimensional normal random vector, independent of S, with zero mean and covariance matrix Σ. t_ν^{-1} denotes the inverse of the univariate t-student distribution function with ν degrees of freedom and $R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$.

The T-copula is radially symmetric and has tail dependency given by

$$\lambda_U = \lambda_L = 2 - 2t_{\nu+1}\left(\frac{(\nu+1)(1-\rho)}{1+\rho}\right)^{1/2}, \tag{2.17}$$

where ρ is the linear correlation of the bivariate t-distribution. The coefficient of upper (lower) tail dependency is increasing in ρ and decreasing in ν. For $\rho < 1$, it tends to zero as the number of degrees of freedom tends to infinity (Embrechts et al. (2001)).

Archimedean copulas

Archimedean copulas constitute an important class of copula functions due to their analytical tractability (many of them have closed form expression), parsimoniously and variety of different dependency structures.

Let us consider a function $\varphi : [0,1] \to [0,\infty]$, continuous, $\varphi'(u) < 0$ for all $u \in [0,1]$ and $\varphi(1) = 0$. We then define pseudo-inverse of φ as the function $\varphi^{[-1]}$: $[0,\infty] \to [0,1]$ such that:

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{[-1]} & \text{for } 0 \leq t \leq \varphi(0) \\ 0 & \text{for } \varphi(0) \leq t \leq \infty \end{cases}$$

If φ is convex, then the function $\mathbf{C} : [0,1]^2 \to [0,1]$ defined as

$$\mathbf{C}(u_1,u_2) = \varphi^{[-1]}[\varphi(u_1) + \varphi(u_2)] \tag{2.18}$$

is an Archimedean copula and φ is called the generator of the copula. Moreover, if $\varphi(0) = \infty$, the pseudo-inverse describes an ordinary inverse function (that is $\varphi^{[-1]} = \varphi^{-1}$) and we call φ and $\mathbf{C}$, respectively, a strict generator and a strict Archimedean copula.

⋆ **Gumbel Copula**. Let $\varphi(t) = (-\ln t)^\theta$ with $\theta \geq 1$ and independency for $\theta = 1$. Then we have

$$\mathbf{C}_\theta^{Gumbel}(u_1, u_2) = \exp\{-[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{1/\theta}\}.$$

⋆ **Clayton Copula**. Let $\varphi(t) = (t^{-\theta} - 1)/\theta$ with $\theta \geq -1$, strict for $\theta \geq 0$ and independency for $\theta = 0$. Then,

$$\mathbf{C}_\theta^{Clayton}(u_1, u_2) = \max[(u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, 0].$$

⋆ **Frank Copula**. Let $\varphi(t) = -\ln \frac{e^{-\theta t}-1}{e^{-\theta}-1}$ with $\theta \in \mathbf{R}\backslash\{0\}$ and independency for $\theta = 0$. Then we obtain the following expression for copula

$$\mathbf{C}_\theta^{Frank}(u_1, u_2) = -\frac{1}{\theta} \ln \left[1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right].$$

For this family of copulas, tail dependency and Kendall's tau coefficients can be expressed in terms of the generator function,[2]

$$\tau = 1 + 4\int_0^1 \frac{\phi(u)}{\phi'(u)} du, \tag{2.19}$$

$$\lambda_U = 2 - 2\lim_{u\to 0} \frac{\phi^{-1'}(2u)}{\phi^{-1'}(u)}, \tag{2.20}$$

$$\lambda_L = 2\lim_{u\to\infty} \frac{\phi^{-1'}(2u)}{\phi^{-1'}(u)}. \tag{2.21}$$

provided that the limits exist.

The Clayton copula has lower tail dependency ($2^{-1/\theta}$) but not upper tail dependency and $\tau = \frac{\theta}{\theta+2}$. The Gumbel copula only has upper tail dependency ($2 - 2^{1/\theta}$) and $\tau = 1 - 1/\theta$. The Frank copula has neither upper nor lower tail dependency and $\tau = 1 + 4\frac{D_1(\theta)-1}{\theta}$, where $D_1(\theta)$ is the first order Debye function, defined as $D_1(\theta) = \frac{1}{\theta}\int_0^\theta \frac{t}{\exp(t)-1}dt$. For these copulas, the larger the parameter θ (in absolute value), the stronger the dependency structure.

[2]See Galiani (2003) and Nelsen (1999).

In the Archimedean copulas framework, the dependency between any two random variables does not depend on which variables we choose. Hence, the parameters do not have a straightforward meaning in multivariate data. In terms of credit risk analysis, this imposes an important restriction on the dependency structure since the default dependency is the same between any set of firms. This is precisely the reason for using pairs of variables in the empirical application.

2.1.4 Calibration

This subsection presents the main methods to calibrate copula parameters proposed in statistical literature.[3] In the following analysis consider a random sample represented by the time series $Y = (Y_{1,t}, Y_{2,t}, \ldots, Y_{N,t})_{t=1}^{T}$, where N is the number of underlying assets (firms) included and T represents the number of observations available on a periodic (daily) basis.

The maximum likelihood (ML) method

Let Θ be the parameter space and θ be the k-dimensional vector of parameters to be estimated. Let $L_t(\theta)$ and $l_t(\theta)$ be, respectively, the likelihood and log-likelihood for the observation at time t. The log-likelihood function $l(\theta)$ is defined as

$$l(\theta) = \Sigma_{t=1}^{T} l_t(\theta). \tag{2.22}$$

Expanding the previous expression using the copula function presented in equation (2.3), we obtain

$$l(\theta) = \Sigma_{t=1}^{T} \ln c(F_1(y_1^t), \ldots, F_N(y_N^t)) + \Sigma_{t=1}^{T}\Sigma_{i=1}^{N} \ln f_i(y_i^t). \tag{2.23}$$

Then, the maximum likelihood estimator is defined as the vector $\hat{\theta}$ such that

$$\hat{\theta} := (\hat{\theta}_1, \ldots, \hat{\theta}_k) \in \arg\max\{l(\theta) : \theta \in \Theta\}.$$

[3] See Romano (2002), Mashal and Zeevi (2002) and Joe and Xu (1996).

As the ML estimates the dependency structure and the margins parameters simultaneously, it is computationally very intensive. Therefore, two alternative methodologies follow.

The inference functions for margins (IFM) method

According to this method,[4] the parameters of marginal distributions are estimated separately from the parameters of the dependency structure. The log-likelihood function $l(\theta)$ presented in equation (2.23) is expressed as follows

$$l(\theta) = \Sigma_{t=1}^{T} \ln c(F_1(y_1^t;\theta_1),\ldots,F_N(y_N^t;\theta_N);\alpha) + \Sigma_{t=1}^{T}\Sigma_{i=1}^{N} \ln f_i(y_i^t;\theta_i). \qquad (2.24)$$

In the above expression, the separation between the vector of the parameters for the univariate marginals $\theta = (\theta_1,\ldots,\theta_N)$ and the vector of the copula parameters α is clear. Summing up, the estimation process is dived into two steps:

1. Estimation of the vector of the parameters for univariate marginals $\theta = (\theta_1,\ldots,\theta_N)$ using ML method. For example, for the $i-th$ underlying asset, we have $\hat{\theta}_i = \arg\max_{\theta_i} \sum_{t=1}^{T} \ln f_i(y_i^t;\theta_i)$.

2. Estimation of the vector of the copula parameters α using the previous estimators $\hat{\theta} = (\hat{\theta}_1,\ldots,\hat{\theta}_N)$:

$$\hat{\alpha}_{IFM} = \arg\max_{\alpha} \Sigma_{t=1}^{T} \ln c(F_1(y_1^t;\hat{\theta}_1),\ldots,F_N(y_N^t;\hat{\theta}_N);\alpha).$$

The IFM estimator[5] is then defined as the vector $\theta^{IFM} = (\hat{\theta},\hat{\alpha}_{IFM})$.

The canonical maximum likelihood (CML) method

Both ML and IFM methods are based on an exogenous imposition of a parametric form of the univariate marginals. The CML differs from the IFM method because

[4] Joe and Xu (1996).

[5] Note that in the case of gaussian multivariate copula with gaussian margins, given that the only parameter to estimate is the correlation matrix, the outputs of ML and IFM are equivalent.

no assumptions are needed about the parametric form of the marginal distributions. This method relies on the concept of *empirical marginal transformation*. This transformation tends to approximate[6] the marginals $\hat{F}_i$, for $i = 1, \ldots, N$, with the empirical distribution functions $\hat{F}_i(\cdot)$ defined as

$$\hat{F}_i(\cdot) = \frac{1}{T}\Sigma_{t=1}^{T} 1_{\{Y_{it} \leq \cdot\}},\ for\ i = 1, \ldots, N, \tag{2.25}$$

where $1_{\{.\}}$ represents the indicator function. The CML method is performed into two steps:

1. Transformation of the initial data set $Y = (Y_{1t}, \ldots, Y_{Nt})_{t=1}^{T}$ into uniform variables, using the empirical marginal distribution described above, *i.e.*, for $t = 1, \ldots, T$, let $\hat{u}_t = (\hat{u}_1^t, \ldots, \hat{u}_N^t) = [\hat{F}_1(Y_{1t}), \ldots, \hat{F}_N(Y_{Nt})]$.

2. Estimation of the vector of copula parameters α using the following relation:

$$\hat{\alpha}_{CML} = \arg\max_{\alpha} \Sigma_{t=1}^{T} \ln c(\hat{u}_1^t, \ldots, \hat{u}_N^t; \alpha).$$

The CML estimator is then defined as the vector $\theta^{CML} = \hat{\alpha}_{CML}$.

2.1.5 The conditional copula

This is an extension of some of the early applications of copulas in statistical modelling where the random vector of interest could be assumed to be independent and identically distributed (iid).

The conditional likelihood is given by the conditional version of the Sklar's Theorem 3. Let F_i be the conditional distribution of $Y_i|W$ for $i = 1, \ldots, N$, W be some information set and F be the joint (absolutely continuous) conditional distribution of $Y|W$, where $Y = (Y_1, \ldots, Y_N)$ has conditional copula function $\mathbf{C}$. Then,

$$F(y_1, \ldots, y_N|w) = \mathbf{C}(F_1(y_1|w), \ldots, F_N(y_N|w)|w).$$

[6]See Mashal and Zeevi (2002).

with the following adaptation of Corollary 5:

$$f(y_1, \ldots, y_N|w) = c(F_1(y_1|w), \ldots, F_N(y_I|w)|w) \prod_{i=1}^{N} f_i(y_i|w), \tag{2.26}$$

where $f_i(y_i|w)$ is the conditional density of $Y_i|W = w$ and

$$c(u_1, \ldots, u_N|w) = \frac{\partial^N C(u_1, \ldots, u_N|w)}{\partial u_1 \ldots \partial u_N}.$$

The log-likelihood expression that is equivalent to the one presented in equation (2.23) is given by:

$$l(\theta) = \Sigma_{t=1}^{T} \ln c(F_1(y_1^t|w_t), \ldots, F_N(y_N^t|w_t)|w_t) + \Sigma_{t=1}^{T}\Sigma_{i=1}^{N} \ln f_i(y_i^t|w_t). \tag{2.27}$$

and we can follow all the previous procedures in order to estimate the parameters.

An important restriction introduced of Sklar's theorem applied to conditional distributions is that the conditioning set, W, must be the same for all marginal distributions and for the copula.

2.2 Applications to finance

Copulas are used in several scientific areas, for instance, mathematics, statistics, biostatistics, operations research, natural sciences, engineering, actuarial science, economics and finance. Nelsen (1999) is one of the standard books of copulas combining the most important theorems and fundamental results about this subject. Following this publication, the interest in applying copula methodology to finance increased substantially. In fact, *in June 2006, finance and actuarial science together contributed 47% of the literature*[7] about copulas. In the financial area, the first research group in copulas, which was headed by Paul Embrechts and Mc Neill, published several very cited papers about using copulas in risk management.[8] At the same time, Li (2000) developed a pioneer work about default correlation that

[7] Genest et al. (2009).
[8] See, for example, Embrechts et al. (2001).

models the random variable "time-until-default" and the survival functions, using a Gaussian copula function. Other important references in the financial area, for mathematical and intuitive framework, are Schönbucher (2003), Duffie and Singleton (2003), Lando (2004) and, more recently, McNeil et al. (2005), which combines copula theory and extreme value theory in the context of credit, market, operational risks and insurance.

Nowadays, copulas are widely used in finance.[9] Indeed, according to the database presented in Genest et al. (2009), the major areas of application in finance are the following, with some references:

⋆ Risk Management: Credit, market, operational risks and risk aggregation. See Embrechts et al. (1999), Li (2000) and McNeil et al. (2005);

⋆ Portfolio Management: Dependency between financial markets, different assets and currencies. See, for example, Patton (2004) and Dias and Embrechts (2003);

⋆ Pricing of derivatives: CDS, CDO and other credit risk products. Some references: Meneguzzo and Vecchiato (2002), Galiani (2003), Bluhm (2003), Cherubini et al. (2004), Hull and White (2006), Hull and White (2007), Hull and White (2008) and Hitier and Huber (2009);

⋆ Risk measurement: value-at-risk (VaR), expected shortfall (conditional VaR) and financial contagion. See, for example, Embrechts et al. (2003) and Palaro and Hotta (2006).

A topic that has captured the attention of finance researchers using copula methods is the study of financial contagion.[10] As copulas contain all the information about the dependency structure of a vector of random variables, they can capture nonlinear dependency among variables and, in particular, in the tails of the distribution. This is of extreme importance in periods of high levels of volatility in which

[9]The website *www.defaultrisk.com* provides a substantial set of free downloadable papers on credit risk and, in particular, contagion and dependency. Some of the references mentioned in this subsection can be found there.

[10]See, for instance, Rodriguez (2007).

a specific institution can lead to problems in other institutions, for instance, during economic crisis. Following this line of thought, this thesis studies a set of dependency measures between US financial institutions in two periods of time, testing a structural break in the summer of 2007.

Chapter 3

Empirical analysis

In our study copula theory is used to model the default dependency between four US financial institutions, namely, JP Morgan Chase & Co., Goldman Sachs Inc., Bank of America Corp. and Citigroup Inc. The modelling is done in the following sequence:

- ⋆ An exploratory data analysis is conducted in section 3.1, where the CDS spreads and the CDS spreads variation were analysed;
- ⋆ In section 3.2 we present the results for the general model. The model selection and estimation is done in two steps:
  - In subsection 3.2.1 we use ARMA-GARCH models to fit each of the CDS spreads variations series.
  - In subsection 3.2.2 five different copulas are tested for the bivariate distribution of each pair of institutions, using the obtained ARMA-GARCH innovations.

The software used for the empirical work was MATLAB R2009a (mainly the Econometrics and Statistics toolboxes).[1]

[1] In order to save space, the MATLAB codes and some figures are not presented in this document. Nevertheless, they are available upon request (claudia.maeg@gmail.com).

3.1 Data and Descriptive Statistics

A CDS is a contract, regulated by International Swaps and Derivatives Association (ISDA), which provides insurance against losses arising to creditors from a firm's default. In a CDS transaction, the spread paid by the CDS buyer to insure against credit events[2] is usually expressed as an annualised percentage of the notional value. As their liquidity have increased a lot in the last years, CDS spreads can be used to capture the closeness to default of a firm, being a more reliable indicator of default than equity prices. When the spread approaches 100%, assuming no recovery rate, the company is on the verge of bankruptcy. Unlike the usual practice of using equity returns to fit the copula,[3] we use CDS spreads, which are a more direct indicator of default dependency between financial institutions.

We analyse the dependency between four US financial institutions, pair by pair, in two periods of time: before the crisis, since 2 January 2006 until 19 July 2007 (404 observations), and during the crisis, since 20 July 2007 until 31 March 2009 (442 observations). The period division on mid-July of 2007 coincides with the first signs of the liquidity crisis, when the losses of some funds and banks with subprime investments forced central banks to extraordinary liquidity interventions. The second period remains until the first shy signs of recovery on economies in March 2009, as explained on chapter 1. The start of the pre-crisis was chosen to be January 2006, to allow that the two subperiods had approximately the same number of observations. We considered the US market, whose financial system was particulary affected, represented by the following institutions: JPMorgan Chase & Co., Bank of America Corp, Citigroup Inc and Goldman Sachs Inc. JP Morgan Chase Bank, Bank of America NA and Citibank, are the top three US banks, 100% owned by the first three abovementioned institutions, respectively. This ranking is based on total assets and is published by BANKERSAlmanac.[4] The choice of

[2]Generally, five events are included: the reference entity fails to meet payment obligations when they are due; bankruptcy; repudiation; material adverse restructuring of debt and acceleration or default of obligation.

[3]See Elizalde (2005).

[4]Available at www.bankersalmanac.com

Goldman Sachs was due to its restructuring during the crisis, as we will see below. The information on the CDS spreads for senior unsecured debt of these institutions was extracted from Markit's CDS database. We consider the spreads of 5-years maturity, as it was the most liquid contract, XR as document clause[5] and US dollar denominated.

Empirical related facts:

- ⋆ JP Morgan Chase have been the institution least affected by the crisis within the considered set of banks, according to its CDS spreads and equity market capitalisation. In March 2008, this institution agreed to purchase Bear Stearns, which was on the verge of bankruptcy, in a transaction sponsored by the US authorities. This acquisition certainly carried some risks, but the degree of public support for this transaction clearly helped to mitigate the impact on JP Morgan market indicators.

- ⋆ Goldman Sachs obtained permission from the US authorities to convert itself into a bank holding company, in September 2008. This new regime, fundamentally, allowed a direct access to FED liquidity providing operations and a sharp reduction in the level of debt, or leverage, of its balance sheet.

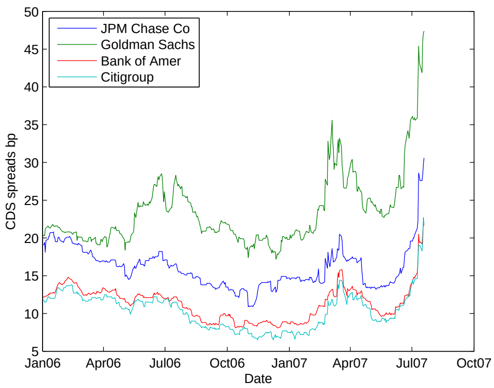
- ⋆ Citigroup was bailed out in November 2008 by three US Federal institutions - the FED, the Treasury Department and the Federal Deposit Insurance Corporation (FDIC), which were forced to guarantee the losses on $306bn of risky assets and injected $20bn of capital into the banking group.

- ⋆ In September 2008 Bank of America Corporation announced its intentions to purchase Merrill Lynch & Co., Inc., in a $50 billion all-stock transaction. The acquisition was supported by the US authorities, through a preferred equity stake and guarantees on a pool of troubled assets, in January 2009.

[5] In these CDS contracts, the credit events that are included in the protection are the standard for high-yield US market, which protects only from failure to pay, bankruptcy or moratorium on an entities debt. Restructuring does not trigger a credit event.

3.1.1 CDS spreads

Figure 3.1 presents the CDS spreads of the four US institutions considered, before and during the crisis. Before the crisis, the CDS spreads ranged between 5 to 50 basis points (bp). Citigroup and Bank of America evolved very close, exhibiting smaller spreads than JP Morgan and Goldman Sachs. During the crisis, the CDS spreads of the four financial institutions increased significantly. Notice, for example, the premium of 6% (600 bp) for bearing the credit risk of Goldman Sachs, reached in the last quarter of 2008, which was roughly equivalent to a speculative-grade bond's yield in the pre-crisis period. Observing the evolution of CDS spreads, we conclude that they were very reactive in the crisis period, with different behaviour along the crisis. In particular, we note the huge volatility in CDS of Goldman Sachs, which was an investment bank with a business profile similar to Lehman Brothers, and Citigroup, due to its successive problems throughout the crisis. The descriptive statistics are presented in tables 3.1 and 3.2. The registered positive skewness and excessive kurtosis points out to a significant density associated to high CDS spreads (in the right tail).



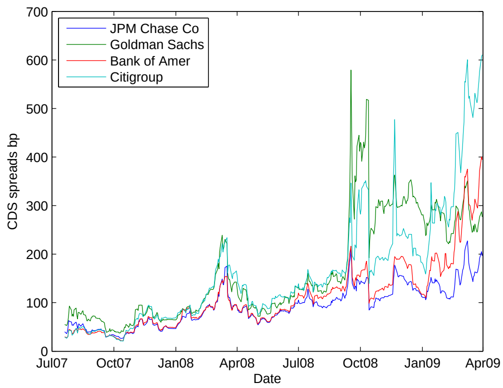


Figure 3.1: CDS spreads before (left-hand side) and during the crisis (right-hand side).

Table 3.1: Summary statistics of CDS spreads before the crisis (bp).

	Mean	Std Dev	Skewness	Kurtosis
JP Morgan	16.06	2.83	1.59	8.13
Goldman Sachs	23.51	4.92	2.04	8.41
Bank of America	11.35	2.28	1.00	5.56
Citigroup	10.36	2.49	0.89	5.41

Table 3.2: Summary statistics of CDS spreads during the crisis (bp).

	Mean	Std Dev	Skewness	Kurtosis
JP Morgan	93.28	41.68	0.52	2.70
Goldman Sachs	164.94	107.70	0.93	3.03
Bank of America	111.01	70.38	1.47	5.61
Citigroup	160.94	121.72	1.55	5.37

Regarding the correlations of the CDS spreads, almost all of them increased in the crisis period. This was expected due to the market panic that usually prevails during the problematic periods (tables 3.3 and 3.4). The top three most correlated pairs, in both periods, are Bank of America vs Citigroup, JP Morgan vs Bank of America and JP Morgan vs Citigroup, respectively. In the pre-crisis period, the correlations among JPMorgan, Citigroup and Bank of America are higher than 0.9 while Goldman Sachs is less correlated with them, with a linear correlation ranging between 0.6 and 0.66, as expected, as among the four institutions, Goldman Sachs was the only investment bank at that moment. During the crisis, the correlation among the other banks maintained a level higher than 0.9 and the correlation levels between Goldman Sachs and the other institutions increased to 0.7-0.9. For this increase may has contributed its conversion into holding bank institution, which result in an approximation of their profile business with the others.

Table 3.3: Correlation matrix of CDS spreads before the crisis (bp).

	JP Morgan	Goldman Sachs	Bank of America	Citigroup
JP Morgan	1.000	0.607	0.912	0.911
Goldman Sachs	0.607	1.000	0.655	0.651
Bank of America	0.912	0.655	1.000	0.982
Citigroup	0.911	0.651	0.982	1.000

Table 3.4: Correlation matrix of CDS spreads during the crisis (bp).

	JP Morgan	Goldman Sachs	Bank of America	Citigroup
JP Morgan	1.000	0.874	0.932	0.907
Goldman Sachs	0.874	1.000	0.772	0.793
Bank of America	0.932	0.772	1.000	0.976
Citigroup	0.907	0.793	0.976	1.000

The behaviour of the CDS spreads over time and the autocorrelation function (see figures 3.2 and 3.3) indicate non-stationarity of the variables before and during the crisis. In fact, according to the Augmented Dickey-Fuller (ADF) test, considering a first order lag of the autoregressive process and the existence of intercept, trend and robust standard errors to heteroscedasticity (White), the hypothesis of a unit root (first order integrated) processes cannot be rejected with 95% confidence (tables 3.5 and 3.6). Therefore, in order to study the default dependency between institutions, the CDS spreads have to be transformed into stationary variables. Its differentiation solved the problem for the generality of the cases, as it will be analysed in the next subsection.

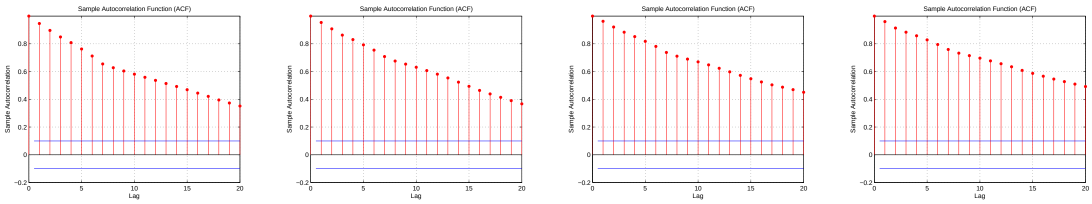


Figure 3.2: ACF of CDS spreads before the crisis (bp), for JPMorgan, Goldman Sachs, Bank of America and Citigroup, respectively.

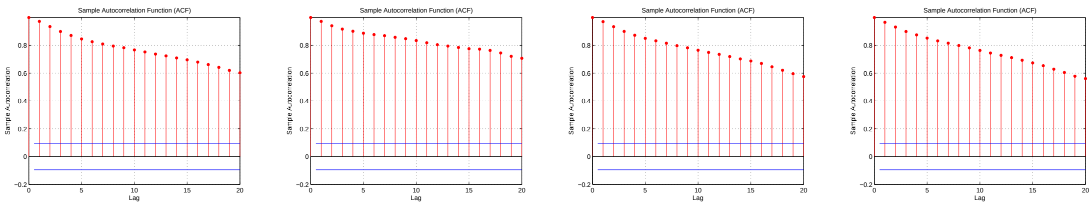


Figure 3.3: ACF of CDS spreads during the crisis (bp), for JPMorgan, Goldman Sachs, Bank of America and Citigroup, respectively.

Table 3.5: ADF test for CDS spreads before the crisis.

	tstat white	critical value 5%	critical value 1%
JP Morgan	1.2285	-3.4225	-3.9839
Goldman Sachs	0.2891	-3.4225	-3.9839
Bank of America	0.9197	-3.4225	-3.9839
Citigroup	0.9156	-3.4225	-3.9839

Table 3.6: ADF test for CDS spreads during the crisis.

	tstat white	critical value 5%	critical value 1%
JP Morgan	-2.4095	-3.4213	-3.9815
Goldman Sachs	-1.3550	-3.4213	-3.9815
Bank of America	-1.2060	-3.4213	-3.9815
Citigroup	-1.3685	-3.4213	-3.9815

3.1.2 CDS spreads variation

The steps followed in the subsection 3.1.1 are repeated for the first differences of the CDS spreads, corresponding to the daily discrete changes of the CDS spreads (or CDS spread variation). The daily variations before the crisis are depicted on the left-hand side of figure 3.4, ranging between -3 bp and 7bp, while on the right-hand side of the figure, during the crisis, we observe daily changes higher than 100 bp. The tails in both periods are very heavy with the skewness changing from positive, in the first period, to negative, in the second period (tables 3.7 and 3.8), indicating non-Gaussian behaviour.

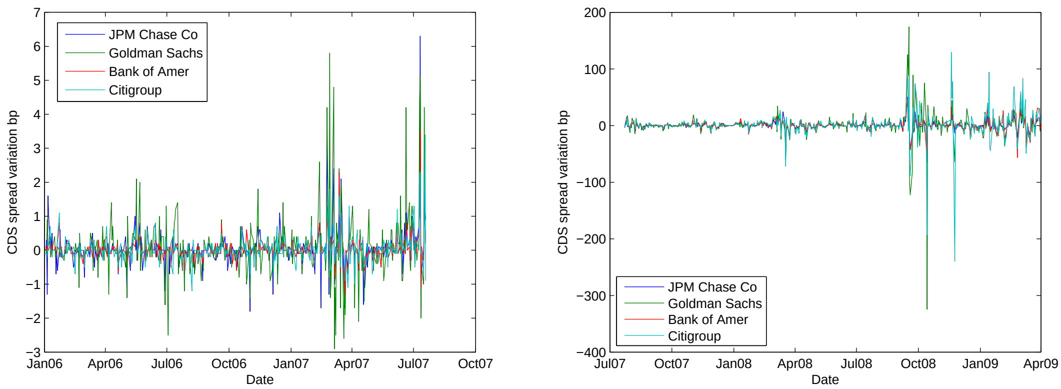


Figure 3.4: CDS spreads variation before (left-hand side) and during the crisis (right-hand side).

Table 3.7: Summary statistics of CDS spreads variation before the crisis (bp).

	Mean	Std Dev	Skewness	Kurtosis
JP Morgan	0.029	0.56	4.00	45.53
Goldman Sachs	0.067	0.89	2.25	15.73
Bank of America	0.024	0.35	3.50	33.68
Citigroup	0.025	0.41	2.73	20.07

Table 3.8: Summary statistics of CDS spreads variation during the crisis (bp).

	Mean	Std Dev	Skewness	Kurtosis
JP Morgan	0.354	8.12	-1.44	22.29
Goldman Sachs	0.502	24.24	-4.61	85.20
Bank of America	0.828	9.82	-0.34	13.11
Citigroup	1.319	22.75	-3.23	47.25

As in the CDS spreads case, the linear correlation among the CDS spreads variation of the financial institutions increased in the crisis period (tables 3.9 and 3.10) for the generality of the institutions, with the pairs Goldman Sachs vs Bank of America and Bank of America vs Citigroup being the exceptions. Before the crisis, the least correlated pair was JPMorgan vs Citigroup (0.530) and the most correlated was Bank of America vs Citigroup (0.679). In the crisis period, the highest correlation occurred between JPMorgan vs Bank of America (0.82) and the smallest (0.555) belongs to the pair Bank of America vs Goldman Sachs. Note that the correlations of CDS spreads (tables 3.3 and 3.4) are higher than the respective correlations of CDS spreads variations, for the generality of the cases. However, recall that the correlation coefficient is very dependent on the extreme observations. Hence, we will return to the association level later, when other alternative measures will be presented within the copula framework.

Table 3.9: Correlation matrix of CDS spreads variation before the crisis (bp).

	JP Morgan	Goldman Sachs	Bank of America	Citigroup
JP Morgan	1.000	0.620	0.615	0.530
Goldman Sachs	0.620	1.000	0.585	0.540
Bank of America	0.615	0.585	1.000	0.679
Citigroup	0.530	0.540	0.679	1.000

Table 3.10: Correlation matrix of CDS spreads variation during the crisis (bp).

	JP Morgan	Goldman Sachs	Bank of America	Citigroup
JP Morgan	1.000	0.739	0.820	0.744
Goldman Sachs	0.739	1.000	0.555	0.687
Bank of America	0.820	0.555	1.000	0.653
Citigroup	0.744	0.687	0.653	1.000

The stationarity of the CDS spreads variations must be checked for the two considered periods. According to Dickey Fuller (DF) test, considering intercept and standard errors robust to heteroscedasticity (White), the hypothesis of a unit root (first order integrated) processes is rejected with 99% confidence (tables 3.11 and 3.12). Hence, we can proceed with the CDS spreads variation as our interest variable of default closeness indicator.

As mentioned in tables 3.7 and 3.8, kurtosis is very high for all banks, suggesting the non-normality of CDS spreads variations. The univariate non-normality of the series was formally tested using the Jarque-Bera test, being rejected at 99% confidence (tables 3.13 and 3.14). The corresponding Quantile-Quantile (QQ) plots, in figures 3.5 and 3.6, depict the significant differences between the quantiles of a Gaussian variable (red dashed line) and the empirical distributions (blue points). Therefore, before proceeding with the modelling of the marginal distributions, we analyse the autocorrelation functions (ACF) and partial autocorrelation functions (PACF) of the CDS spreads variation for the considered institutions, before and during the crisis (figures 3.7, 3.8, 3.9 and 3.10). The first lag of PACF is statistically significant at 95% in some cases. Furthermore, figure 3.4 suggests the existence of time-varying variance. The ARCH test (tables 3.15 and 3.16) confirms this fact, rejecting the null hypothesis of non-existence of conditional heteroscedasticity until the lag 20 with 90% confidence. This holds for the generality of the cases, with JP Morgan and Citigroup being the exceptions. The first one in the pre-crisis and the second one in the crisis period, respectively. In line with these conclusions and for the sake of consistency, the same GARCH specification for conditioned heteroscedasticity was implemented for all banks and both periods.

3.2 Empirical results

As previously mentioned, a copula joins a group of univariate marginal distributions functions by using, in our case, CDS data. The behaviour of the univariate marginal distributions is established using a time-varying ARMA-GARCH specification for the CDS spreads variation. Then, the input of the copula correspond to the t-cumulative distribution function of the ARMA-GARCH standardized residuals. As these residuals are used a proxy for default closeness, we are able to determine the implied dependency structure, providing information on the contagion risk between financial institutions.

Table 3.11: DF test for CDS spreads variation before the crisis.

	tstat white	critical value 5%	critical value 1%
JP Morgan	-12.4683	-2.8694	-3.4491
Goldman Sachs	-9.5688	-2.8694	-3.4491
Bank of America	-6.6497	-2.8694	-3.4491
Citigroup	-7.9684	-2.8694	-3.4491

Table 3.12: DF test for CDS spreads variation during the crisis.

	tstat white	critical value 5%	critical value 1%
JP Morgan	-11.8925	-2.8688	-3.4464
Goldman Sachs	-6.3415	-2.8688	-3.4464
Bank of America	-9.3081	-2.8688	-3.4464
Citigroup	-9.8767	-2.8688	-3.4464

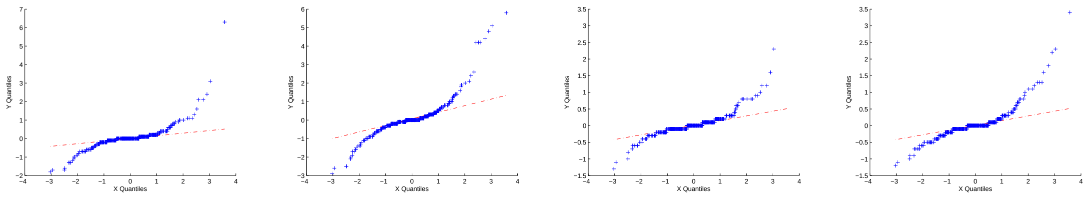


Figure 3.5: QQ-plots of the normal versus the empirical quantiles of CDS spreads variation before the crisis (bp), for JPMorgan, Goldman Sachs, Bank of America and Citigroup, respectively.

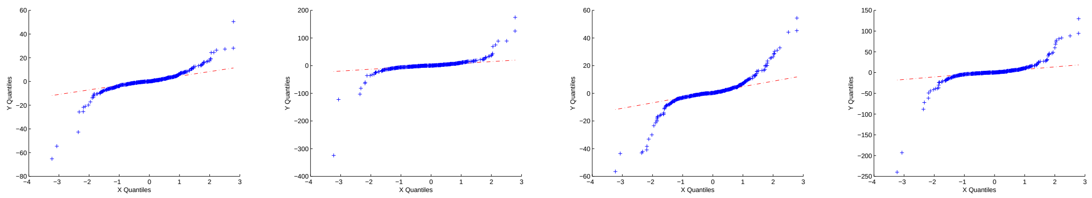


Figure 3.6: QQ-plots of the normal versus the empirical quantiles of CDS spreads variation during the crisis (bp), for JPMorgan, Goldman Sachs, Bank of America and Citigroup, respectively.

Table 3.13: Jarque-Bera test for CDS spreads variation before the crisis.

	Test Statistic	P-value
JP Morgan	31447	0.001
Goldman Sachs	3060	0.001
Bank of America	16626	0.001
Citigroup	5395	0.001

Table 3.14: Jarque-Bera test for CDS spreads variation during the crisis.

	Test Statistic	P-value
JP Morgan	6989	0.001
Goldman Sachs	125728	0.001
Bank of America	1887	0.001
Citigroup	36748	0.001

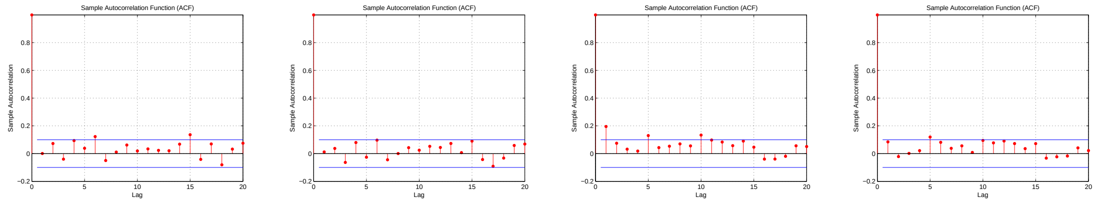


Figure 3.7: ACF of CDS spreads variation before the crisis (bp), for JPMorgan, Goldman Sachs, Bank of America and Citigroup, respectively.

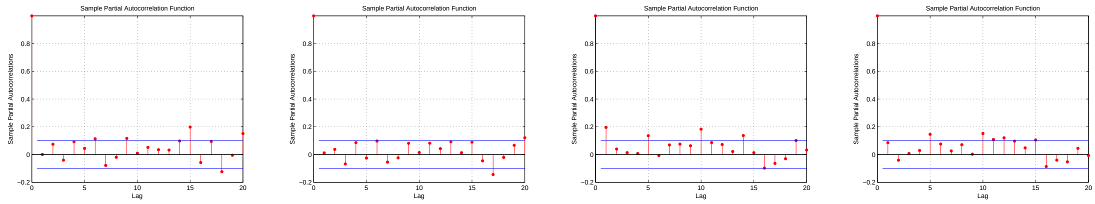


Figure 3.8: PACF of CDS spreads variation before the crisis (bp), for JPMorgan, Goldman Sachs, Bank of America and Citigroup, respectively.

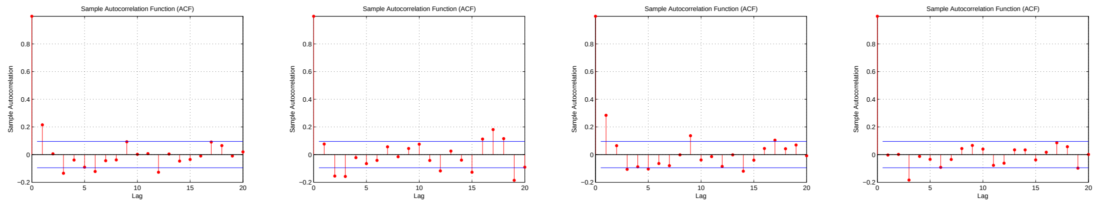


Figure 3.9: ACF of CDS spreads variation during the crisis (bp), for JPMorgan, Goldman Sachs, Bank of America and Citigroup, respectively.

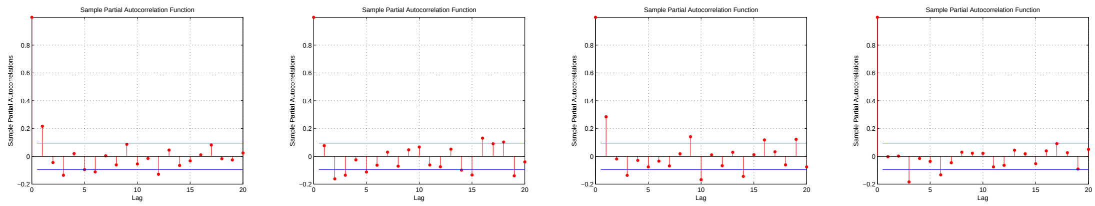


Figure 3.10: PACF of CDS spreads variation during the crisis (bp), for JPMorgan, Goldman Sachs, Bank of America and Citigroup, respectively.

Table 3.15: ARCH test for CDS spreads variation before the crisis.

	ARCH Statistic	P-value
JP Morgan	12	0.910
Goldman Sachs	83	0.000
Bank of America	43	0.000
Citigroup	163	0.000

Table 3.16: ARCH test for CDS spreads variation during the crisis.

	ARCH Statistic	P-value
JP Morgan	31	0.052
Goldman Sachs	38	0.008
Bank of America	108	0.000
Citigroup	20	0.440

3.2.1 Marginal densities

According to the tests performed in section 3.1, the first differences of CDS spreads present time-varying conditional variance and heavy tails. Hence, univariate ARMA-GARCH models are fitted for the CDS spreads variations of each institution, assuming that the innovations come from a t-student distribution (which displays polynomial decay in tails).[6]

The model

The selected model for all banks was ARMA(1,0)-GARCH(1,1). Formally,

$$
\begin{aligned}
X_{i,t} &= \mu_i + \phi_i X_{i,t-1} + \epsilon_{i,t} \\
\epsilon_{i,t} &= \sigma_{i,t} Z_{i,t} \\
\sigma^2_{i,t} &= \gamma_i + \alpha_i \epsilon^2_{i,t-1} + \beta_i \sigma^2_{i,t-1}
\end{aligned}
$$

where $\gamma_i > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $Z_{i,t}$ is independent of $(X_{i,s})_{s \leq t}$ and ν denotes the number of degrees of freedom of the t-distributions assumed for the residuals in the maximum likelihood estimation, for all $i = 1, 2, 3, 4$ banks.

The stationarity conditions are $|\phi_i| < 1$ and $\alpha_i + \beta_i < 1$ for the conditional mean and conditional variance equations, respectively. Occasionally, in the context of financial variables, the model is an Integrated GARCH (IGARCH), *i.e.*, $\alpha_i + \beta_i = 1$ as ϵ^2_t has a unit root. Although ϵ_t is not covariance stationary, it is strictly stationary. The covariance stationarity assumes the existence of the second moment, while the stationarity in strict sense implies that all the probabilistic structure is stable over time, regardless of the moments are finite or not.[7]

[6] The same specifications assuming normality of innovations were also estimated but underperformed the results assuming t-distributions, according to the Akaike (AIC) criterion.

[7] The necessary and sufficient condition for ϵ_t be strictly stationary is $E[log(\beta + \alpha z^2_t)] < 0$, see Nelson (1990).

Modelling the marginal distributions

ARMA(1,0)-GARCH(1,1) was the implemented model, as it exhibited a good performance for the generality of the banks, before and during the crisis. Although some coefficients are not statistically significant at 10% significance level, other ARMA-GARCH specifications (including asymmetric) were also estimated, but without significant improvement. From the fitted ARMA-GARCH parameters, the (iid) standardised residuals $\hat{z}_{i,t} = \hat{\epsilon}_{i,t}/\hat{\sigma}_{i,t}$ are recovered for the CDS spread variation of each bank.

The parameters and the corresponding standard errors are displayed in tables 3.17 and 3.18, before and during the crisis, respectively. The fitted t-distributions have infinite kurtosis, as $\hat{\nu} \leq 4$. The constants of the conditional mean equations ($\hat{\mu}$) are not statistically significant at 90% confidence, before and during the crisis. In the first period, the autoregressive parameter is significant only in the case of Bank of America. However, during the crisis, these estimates increased and became significant. All the institutions have positive autoregressive estimators (less than one to ensure stationarity), with the highest belonging to Bank of America in the both periods. This suggests that the actual CDS spread variation depends positively on the change registered on the day before. Regarding the conditional variance equation, the ARCH and GARCH parameters are all positive and significant for the generality of the cases, before and during the crisis. Note that, for all the cases, the sum of the ARCH and GARCH coefficients is one, indicating IGARCH models for the conditional variances. As already mentioned, this process is not stationary in covariance but it is strictly stationary, *i.e.*, the first and second unconditional moments do not exist but the probability distribution does not change when shifted in time (is stable).

The Ljung-Box test applied to the standardised residuals, as well as to their absolute value, does not reject the hypothesis of null autocorrelations from lag 1 to lag 20, at the 5% significance levels, during the crisis, and, for the generality of the cases, in the pre-crisis period (tables 3.19 and 3.20). Before modelling the dependencies

Table 3.17: ARMA-GARCH models results, estimates (standard errors) for CDS spreads variation before the crisis.

Parameter	JP Morgan	Goldman Sachs	Bank of America	Citigroup
$\hat{\mu}$	0.009 (0.011)	-0.013 (0.019)	-0.007 (0.008)	-0.014 (0.010)
$\hat{\phi}$	0.026 (0.039)	0.037 (0.038)	0.158 (0.041)	0.005 (0.038)
$\hat{\gamma}$	0.039 (0.040)	0.024 (0.014)	0.006 (0.004)	0.008 (0.007)
$\hat{\beta}$	0.764 (0.080)	0.830 (0.045)	0.866 (0.056)	0.908 (0.046)
$\hat{\alpha}$	0.236 (0.238)	0.170 (0.083)	0.134 (0.084)	0.092 (0.066)
$\hat{\nu}$	2.189 (0.205)	2.508 (0.298)	2.512 (0.373)	2.297 (0.230)

Table 3.18: ARMA-GARCH models results, estimates (standard errors) for CDS spreads variation during the crisis.

Parameter	JP Morgan	Goldman Sachs	Bank of America	Citigroup
$\hat{\mu}$	0.173 (0.171)	0.298 (0.274)	0.208 (0.165)	0.281 (0.229)
$\hat{\phi}$	0.203 (0.048)	0.110 (0.047)	0.260 (0.042)	0.157 (0.045)
$\hat{\gamma}$	3.395 (1.163)	4.640 (1.783)	1.984 (0.796)	7.007 (2.180)
$\hat{\beta}$	0.587 (0.075)	0.731 (0.041)	0.721 (0.048)	0.558 (0.048)
$\hat{\alpha}$	0.413 (0.122)	0.269 (0.073)	0.279 (0.086)	0.442 (0.111)
$\hat{\nu}$	3.184 (0.402)	3.118 (0.393)	2.987 (0.371)	2.941 (0.269)

of standardised residuals with copulas, their conditional homoscedasticity must also be checked. The performed ARCH test cannot reject the null hypothesis of non-existence of conditioned heteroscedasticity until the lag 20 for almost all cases, at 95% of confidence (tables 3.21 and 3.22), with the exceptions being Goldman Sachs and Citigroup before the crisis. Furthermore, if the models were correctly specified, the empirical distribution of standardised residuals would be standard uniform series. During the crisis the models fitted really good, as depicted in figure 3.11. Although the ARMA-GARCH specification for JPMorgan and Citigroup do not fit so well before the crisis, they proved to be the most suitable models for the crisis period. Though, for the sake of consistency and given that our priority is the crisis period, the models are considered adequate, according to the previous results.

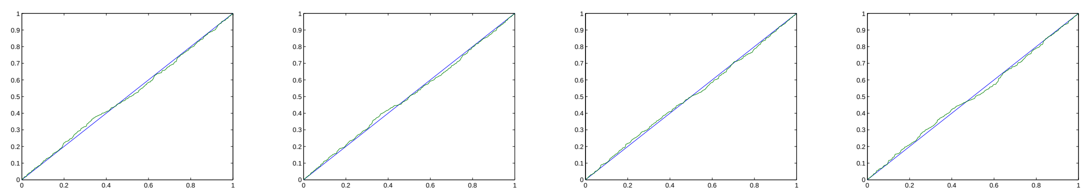


Figure 3.11: Empirical distributions of standardised residuals, during the crisis. From the left-hand side to the right-hand side, JP Morgan, Goldman Sachs, Bank of America and Citigroup.

Table 3.19: Ljung-Box test applied to the standardised residuals before the crisis.

	Ljung-Box Statistic z	P-value	Ljung-Box Statistic $\|z\|$	P-value
JP Morgan	27.22	0.129	18.37	0.563
Goldman Sachs	17.97	0.590	22.02	0.339
Bank of America	31.68	0.047	24.43	0.224
Citigroup	24.09	0.239	32.46	0.039

Table 3.20: Ljung-Box test applied to the standardised residuals during the crisis.

	Ljung-Box Statistic z	P-value	Ljung-Box Statistic $\|z\|$	P-value
JP Morgan	16.73	0.671	9.789	0.972
Goldman Sachs	13.11	0.873	18.32	0.566
Bank of America	21.20	0.386	16.21	0.704
Citigroup	12.61	0.894	9.416	0.978

Table 3.21: ARCH test for standardised residuals before the crisis.

	ARCH Test Stand Res	P-value
JP Morgan	9.23	0.980
Goldman Sachs	39.79	0.005
Bank of America	27.31	0.127
Citigroup	45.42	0.001

Table 3.22: ARCH test for standardised residuals during the crisis.

	ARCH Test Stand Res	P-value
JP Morgan	1.37	1.000
Goldman Sachs	3.34	1.000
Bank of America	9.82	1.000
Citigroup	0.81	1.000

During the crisis, in particular, there is no evidence against serial independency of the standardised residual values in the generality of the cases, according to the cross-correlation depicted in figure 3.12.[8] In fact, only the contemporaneous cross-dependency remains for almost pairs (see lag zero in the same figure), which is exactly where our interest lies. Moreover, we observe a significant increase of the linear correlation of the standardized residuals for all banks, in which the coefficient for the pair JPMorgan vs Citigroup duplicated in the second period (tables 3.23 and 3.24). Compare the tables 3.9 and 3.10 with these correlation matrixes and notice that the coefficients have roughly the same magnitude, but the previous ones depend on heterocedasticity. The positive dependency and the increase in association level

[8]The cross-correlograms are similar before the crisis.

is also evident from the plots of bivariate standardised residual series, before and during the crisis, as presented in figures 3.13 and 3.14, respectively.

Figures 3.15, 3.16, 3.17 and 3.18 depict, on the left-hand side, the 3-D histograms of standardised residuals, for the pairs of institutions JPMorgan vs Citigroup and Goldman Sachs vs Bank of America, before and during the crisis. In the right-hand side of the aforementioned figures are plotted the histograms of the standardised residuals after the transformation into uniform variables, through the cumulative t-distribution function. Note that the standardised residuals (left-hand side) take a wider range of values during the crisis than before the crisis, as it was expected due to the higher variability. It is interesting to note the differences in the tails of the histograms of cumulative standardised residuals, comparing the two periods, for each pair of institutions. As the remaining plots for the other pairs are quite similar to these ones they are not depicted. In the subsection 3.2.2, copula functions will provide a model for the histograms presented on the right-hand side of the abovementioned figures.

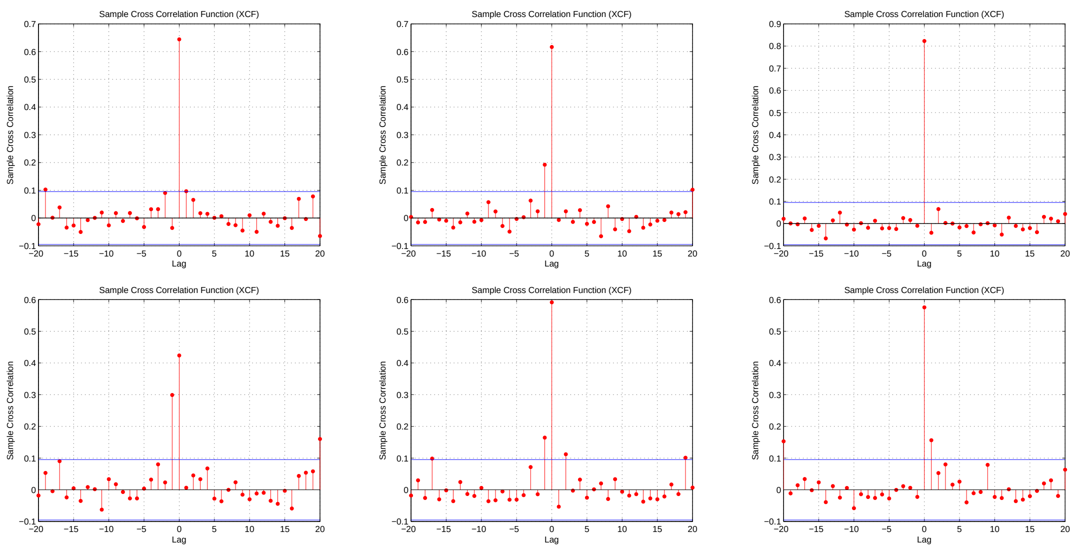


Figure 3.12: Cross-Correlograms of standardised residuals, during the crisis. Above, from the left to the right: JPMorgan vs Goldman Sachs, JPMorgan vs Bank of America, JP-Morgan vs Citigroup. Below: Goldman Sachs vs Bank of America, Goldman Sachs vs Citigroup and Bank of America vs Citigroup.

Table 3.23: Linear correlation of the standardised residuals before the crisis

	JP Morgan	Goldman Sachs	Bank of America	Citigroup
JP Morgan	1.000	0.471	0.470	0.425
Goldman Sachs	0.471	1.000	0.454	0.411
Bank of America	0.470	0.454	1.000	0.540
Citigroup	0.425	0.411	0.540	1.000

Table 3.24: Linear correlation of the standardised residuals during the crisis

	JP Morgan	Goldman Sachs	Bank of America	Citigroup
JP Morgan	1.000	0.737	0.734	0.835
Goldman Sachs	0.737	1.000	0.585	0.654
Bank of America	0.734	0.585	1.000	0.659
Citigroup	0.835	0.654	0.659	1.000

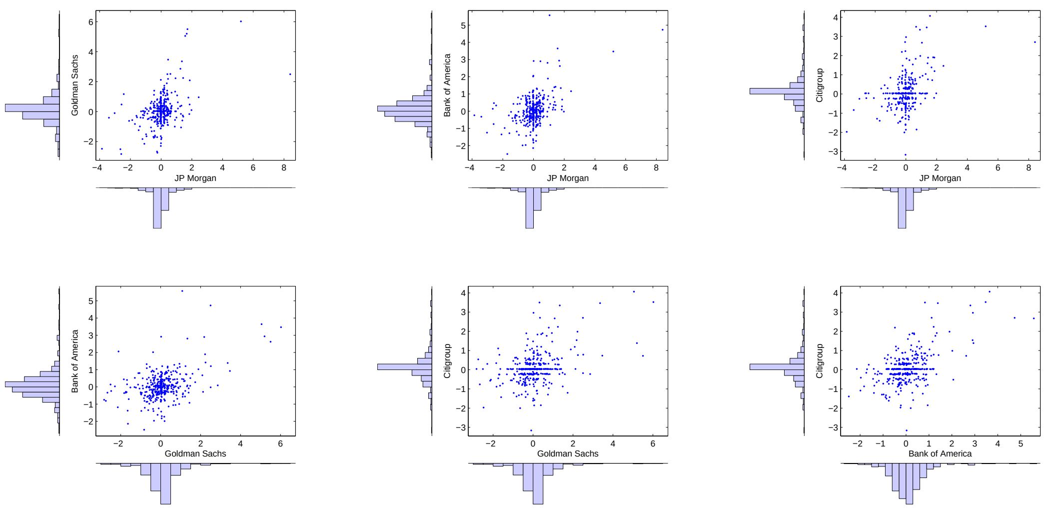


Figure 3.13: Scatter plot of standardised residuals and the individual histograms, before the crisis, for each pair of institutions.

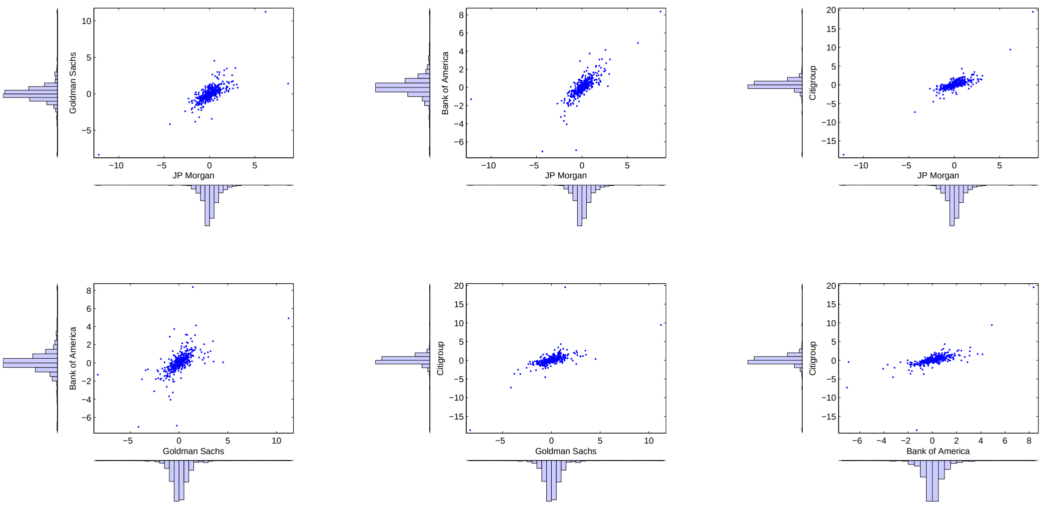


Figure 3.14: Scatter plot of standardised residuals and the individual histograms, during the crisis, for each pair of institutions.

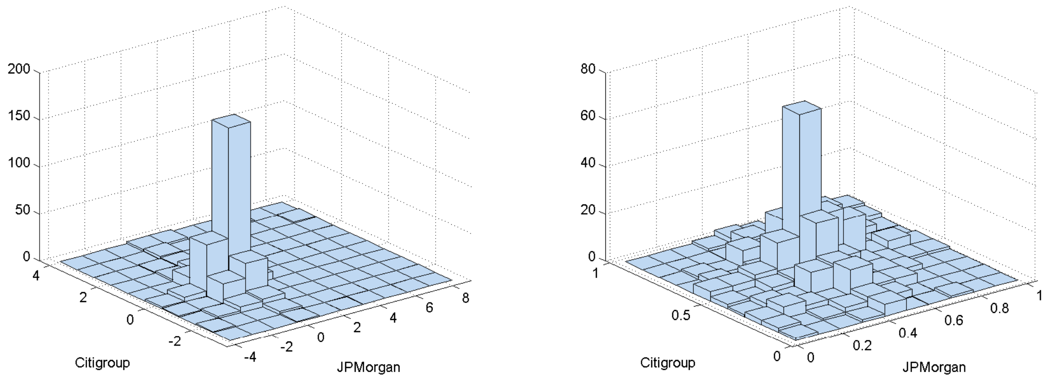


Figure 3.15: 3-D Histograms for the standardised residuals and for the corresponding uniform variables, for JPMorgan vs Citigroup, before the crisis.

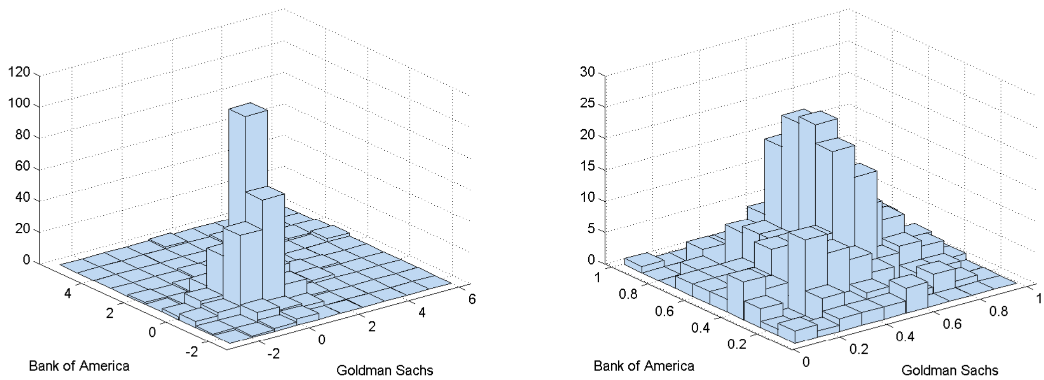


Figure 3.16: 3-D Histograms for the standardised residuals and for the corresponding uniform variables, for Goldman Sachs vs Bank of America, before the crisis.

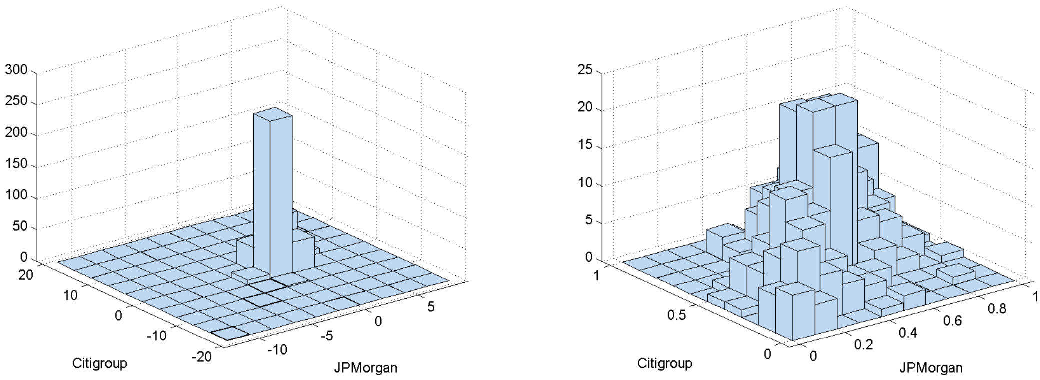


Figure 3.17: 3-D Histograms for the standardised residuals and for the corresponding uniform variables, for JPMorgan vs Citigroup, during the crisis.

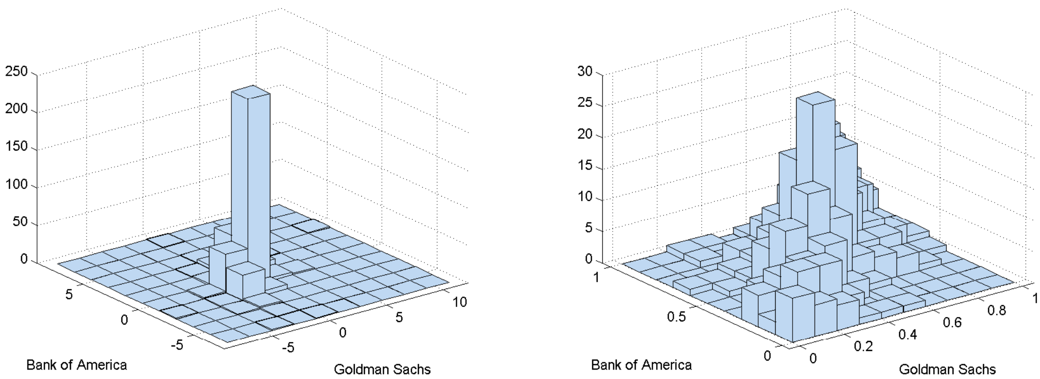


Figure 3.18: 3-D Histograms for the standardised residuals and for the corresponding uniform variables, for Goldman Sachs vs Bank of America, during the crisis.

3.2.2 Joint distributions using conditional copulas

Although the dynamic behaviour of the margins, through ARMA-GARCH modelling, we assume a time-invariant dependency structure within each period, by using static copulas. As we will see below, two different static copulas are selected for the two periods considered, for all the pairs of institutions. Hence, a dynamic copula for the whole period, with a time-varying parameter, would not consider the change in the dependency structure before and during the crisis, which reflects the structural break between the two periods.[9]

Modelling the dependency between institutions

Following the conditional copulas framework (subsection 2.1.5), after the individual ARMA-GARCH modelling, the standardised residuals were recovered and their dependency structure was analysed. The copulas were calibrated using the Inference Functions for Margins (IFM) method described on section 2.1.4, with the input being the standardised residuals (of the CDS spreads variations) transformed into uniform variables, through t-cumulative distribution function.[10] The results are presented by pairs of institutions due to the important restriction imposed by Archimedean copulas, which state that the dependency is the same between any set of firms.

The results regarding the Gaussian and T-copulas, before and during the crisis, are presented in tables 3.25 and 3.26, respectively. The correlations obtained with the Gaussian and the T distributions increased during the crisis, as expected, with the firsts presenting smaller values than the lasts within each period. For the pairs JPMorgan vs Goldman Sachs and JPMorgan vs Bank of America, the degrees of freedom are not statistically significant (at 95% confidence level) in the pre-crisis period. The estimates and the respective standard errors[11] for the different Archimedean copulas estimated are depicted in tables 3.27 and 3.28. All the

[9] See appendix A for a time-varying copula example.

[10] ARMA-GARCH models assuming Gaussian innovations, combined with the empirical distribution function to transform the standardised residuals into uniform variables (CML method, described on subsection 2.1.4) were also tested, but underperformed the IFM method with t-distributed residuals, according to the AIC results.

[11] The covariance matrix of $\hat{\theta}_{IFM}$ was estimated by the inverse of the negative Hessian.

parameters are significant at 99% confidence level. As in the case of the elliptic copulas, the parameters and, consequently, the dependency increased substantially, for the all pairs of institutions under analysis.

Table 3.25: Gaussian and T parameters for each pair of banks, before the crisis.

Parameters	Gaussian: ρ	T: ρ	T: ν (std errors)
JPMorgan vs Goldman Sachs	0.4321	0.7803	9.2209 (7.1276)
JPMorgan vs Bank of America	0.4094	0.7846	3.1619 (2.1819)
JPMorgan vs Citigroup	0.4016	0.7717	1.7203 (0.6077)
Goldman Sachs vs Bank of America	0.3876	0.5857	1.2941 (0.3194)
Goldman Sachs vs Citigroup	0.3654	0.6656	1.7532 (0.5239)
Bank of America vs Citigroup	0.4831	0.7027	1.1609 (0.4088)

Table 3.26: Gaussian and T parameters for each pair of banks, during the crisis.

Parameters	Gaussian: ρ	T: ρ	T: ν (std errors)
JPMorgan vs Goldman Sachs	0.7473	0.8619	7.2329 (3.1862)
JPMorgan vs Bank of America	0.8138	0.9077	3.9756 (1.0458)
JPMorgan vs Citigroup	0.7561	0.8625	5.9893 (2.1770)
Goldman Sachs vs Bank of America	0.6896	0.8518	3.5590 (0.8576)
Goldman Sachs vs Citigroup	0.7070	0.8420	3.9309 (1.1133)
Bank of America vs Citigroup	0.7551	0.8742	4.1893 (1.1935)

Table 3.27: Archimedean copula parameters (std errors) for each pair of banks, before the crisis.

θ	Frank	Gumbel	Clayton
JPMorgan vs Goldman Sachs	7.0669 (0.4136)	2.1921 (0.0982)	2.0052 (0.1325)
JPMorgan vs Bank of America	7.4661 (0.4148)	2.4010 (0.1033)	2.1605 (0.1332)
JPMorgan vs Citigroup	8.0106 (0.3980)	2.4920 (0.0984)	2.2507 (0.1286)
Goldman Sachs vs Bank of America	6.2676 (0.3696)	2.0763 (0.0893)	1.7027 (0.1266)
Goldman Sachs vs Citigroup	6.4741 (0.3688)	2.0507 (0.0854)	1.7909 (0.1235)
Bank of America vs Citigroup	7.8505 (0.4070)	2.5288 (0.1054)	2.4794 (0.1448)

Table 3.28: Archimedean copula parameters (standard errors) for each pair of banks, during the crisis.

θ	Frank	Gumbel	Clayton
JPMorgan vs Goldman Sachs	9.8888 (0.4607)	2.6721 (0.0916)	3.2155 (0.1414)
JPMorgan vs Bank of America	12.3194 (0.5233)	3.3871 (0.1131)	3.8045 (0.1137)
JPMorgan vs Citigroup	9.8005 (0.4469)	2.7286 (0.0997)	3.0614 (0.1390)
Goldman Sachs vs Bank of America	9.9103 (0.4243)	2.5874 (0.0778)	2.6928 (0.0990)
Goldman Sachs vs Citigroup	9.2593 (0.4122)	2.5291 (0.0766)	2.8179 (0.1338)
Bank of America vs Citigroup	10.8262 (0.4543)	2.8621 (0.0948)	3.0154 (0.0962)

Robustness of the copula estimates

The adopted criterion to choose the copula is the Akaike's information criterion (AIC), defined by

$$AIC(M) = -2\text{log-likelihood}(\widehat{\theta}^{IFM}) + 2M, \tag{3.1}$$

where M is the number of parameters being estimated, assuming that the models for marginal distributions are known, and $\widehat{\theta}^{IFM}$ denotes the maximum likelihood estimates according to the IFM method. The smaller the AIC values, the more accurate is the model.

Table 3.29: AIC criterion for each pair of banks, before the crisis.

AIC	Gaussian	T	Frank	Gumbel	Clayton
JPMorgan vs Goldman Sachs	-79.77	-179.40	-175.32	-160.61	-132.53
JPMorgan vs Bank of America	-71.60	-201.85	-189.55	-194.72	-137.02
JPMorgan vs Citigroup	-68.70	-216.13	-211.91	-204.51	-139.34
Goldman Sachs vs Bank of America	-63.65	-164.53	-134.64	-147.16	-100.02
Goldman Sachs vs Citigroup	-55.69	-155.47	-140.58	-134.57	-104.06
Bank of America vs Citigroup	-105.46	-246.02	-218.43	-232.98	-174.91

Table 3.30: AIC criterion for each pair of banks, during the crisis.

AIC	Gaussian	T	Frank	Gumbel	Clayton
JPMorgan vs Goldman Sachs	-358.77	-419.62	-429.67	-361.80	-386.90
JPMorgan vs Bank of America	-477.35	-572.84	-573.47	-535.43	-450.10
JPMorgan vs Citigroup	-372.26	-433.87	-423.03	-390.68	-363.80
Goldman Sachs vs Bank of America	-283.01	-374.58	-417.65	-323.45	-292.52
Goldman Sachs vs Citigroup	-303.45	-377.48	-382.06	-321.86	-330.24
Bank of America vs Citigroup	-370.84	-449.04	-477.03	-410.14	-336.04

Tables 3.29 and 3.30 present the AIC values for each pair of institutions and for all the copulas, before and during the crisis, respectively. As Gaussian and Clayton copulas have higher AIC values than all the other copulas in both periods, they were discarded. According to this criterion, for the generality of institutions, the copulas that minimise the AIC criterion are the T and Frank, before and during the crisis, respectively. However, as already has been pointed out, the estimates of the degrees of freedom for some pairs of banks are not statistically significant. Therefore, in table 3.33, we present the corresponding confidence intervals (at 95%

confidence level),[12] giving strength to the inadequacy of the T-copula for the first two pairs before the crisis. As an alternative for the T-copula, according to AIC values, Gumbel or Frank can be chosen depending on the pairs of institutions. During the crisis, only for the pair JPMorgan vs Citigroup, the AIC criterion for the T-copula is marginally better than the corresponding for the Frank copula. Thus, choosing the Frank copula for all pairs in the crisis period does not carry a significant error. Moreover, the global dependency measures are relatively robust to the copula choice in each period, being of the same magnitude (tables 3.31 and 3.32).

Table 3.31: Rank correlation measures for T, Frank and Gumbel copulas, before the crisis.

	Kendall's tau			Spearman's rho		
	T	Frank	Gumbel	T	Frank	Gumbel
JP Morgan vs Goldman Sachs	0.570	0.565	0.544	0.765	0.766	0.731
JP Morgan vs Bank America	0.574	0.582	0.584	0.770	0.783	0.772
JP Morgan vs Citigroup	0.561	0.603	0.599	0.757	0.804	0.787
Goldman Sachs vs Bank America	0.398	0.528	0.518	0.568	0.726	0.703
Goldman Sachs vs Citigroup	0.464	0.538	0.512	0.648	0.737	0.696
Bank America vs Citigroup	0.496	0.597	0.605	0.686	0.798	0.793

Table 3.32: Rank correlation measures for T, Frank and Gumbel copulas, during the crisis.

	Kendall's tau			Spearman's rho		
	T	Frank	Gumbel	T	Frank	Gumbel
JP Morgan vs Goldman Sachs	0.661	0.663	0.626	0.851	0.858	0.813
JP Morgan vs Bank America	0.724	0.719	0.705	0.900	0.901	0.880
JP Morgan vs Citigroup	0.662	0.660	0.634	0.852	0.856	0.820
Goldman Sachs vs Bank America	0.649	0.663	0.614	0.840	0.858	0.801
Goldman Sachs vs Citigroup	0.637	0.645	0.605	0.830	0.842	0.793
Bank America vs Citigroup	0.677	0.687	0.651	0.864	0.877	0.835

If the same copula is considered before and during the crisis, the estimated parameters for the subperiods would be totally different for the two best Archimedean copulas. The last two columns of table 3.33, which display the confidence intervals for estimates of the copula functions Frank and Gumbel, illustrate this point. For example, the confidence intervals for Gumbel copula applied to the pair JPMorgan vs Bank of America are $[2.20; 2.60]$ and $[3.17; 3.61]$, before and during the crisis, respectively. For almost all pairs, the confidence intervals for both periods are disjoint,

[12]The confidence intervals are calculated like usually, given some significance level α %, by $\hat{\theta} \pm$std error$\times \Phi^{-1}(1-\alpha/2)$, where Φ^{-1} is the inverse of the standard gaussian distribution.

regarding to the Frank and Gumbel copulas, with the exceptions being JPMorgan vs Citigroup and Bank of America vs Citigroup for Gumbel copula. Hence, there is statistical evidence of different parameters in the two sub-periods. This finding supports the idea that the dependency (level) changed significantly in the summer of 2007. Furthermore, the adequacy of two different copulas T and Frank (according to AIC criterion) confirms definitely the existence of a structural break among the pre-crisis and the crisis periods, with the exception being the pair JPMorgan vs Citigroup, for which the selected specification is the T-copula for both periods.

Table 3.33: Confidence intervals for the estimates of the degrees of freedom (ν) of the T-copula and for the parameters of Gumbel and Frank copulas, before and during the crisis.

		T		Frank		Gumbel	
		Before	During	Before	During	Before	During
JPMorgan vs Goldman Sachs	L	-4.75	0.99	6.26	8.99	2.00	2.49
	U	23.19	13.48	7.88	10.79	2.38	2.85
JPMorgan vs Bank America	L	-1.11	1.93	6.65	11.29	2.20	3.17
	U	7.44	6.03	8.28	13.35	2.60	3.61
JPMorgan vs Citigroup	L	0.53	1.72	7.23	8.92	2.30	2.53
	U	2.91	10.26	8.79	10.68	2.68	2.92
Goldman Sachs vs Bank America	L	0.67	1.88	5.54	9.08	1.90	2.43
	U	1.92	5.24	6.99	10.74	2.25	2.74
Goldman Sachs vs Citigroup	L	0.73	1.75	5.75	8.45	1.88	2.38
	U	2.78	6.11	7.20	10.07	2.22	2.68
Bank America vs Citigroup	L	0.36	1.85	7.05	9.94	2.32	2.68
	U	1.96	6.53	8.65	11.72	2.74	3.05

Figures 3.19, 3.20, 3.21 and 3.22 display the logarithm of the probability density function, on the left-hand side, and the cumulative density function, on the right-hand side, for two pairs of institutions, JPMorgan vs Goldman Sachs and Bank of America vs Citigroup.[13] In the first two figures we present the distributions for the period before the crisis, using the T copula, while the last two figures depict the Frank copula in the crisis period. Recall the right-hand side of the figures 3.15, 3.16, 3.17 and 3.18, where the 3D histograms for the standardised residuals transformed into uniform variables can be directly compared with the copula density functions, which provide a model for the information plotted in the histograms.

[13]The remaining pairs have quite similar functions, as we choose the same copula specification.

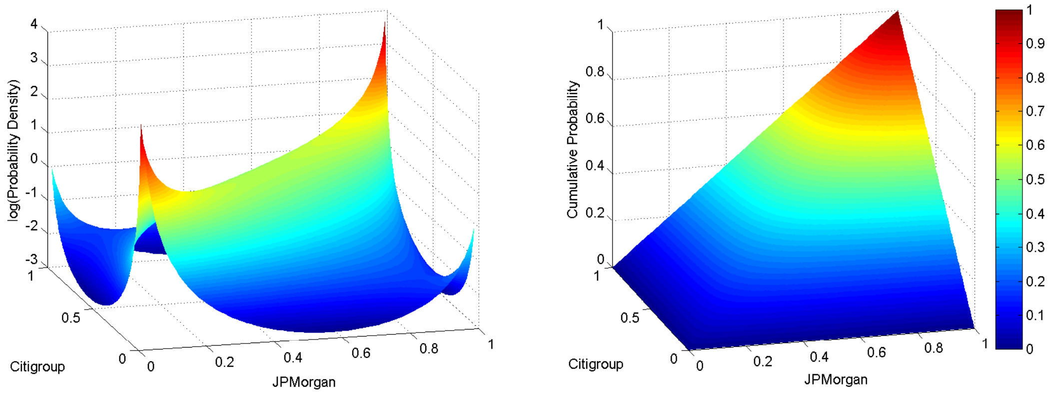


Figure 3.19: Probability and cumulative distribution functions, for JP Morgan vs Citigroup, before the crisis, T copula.

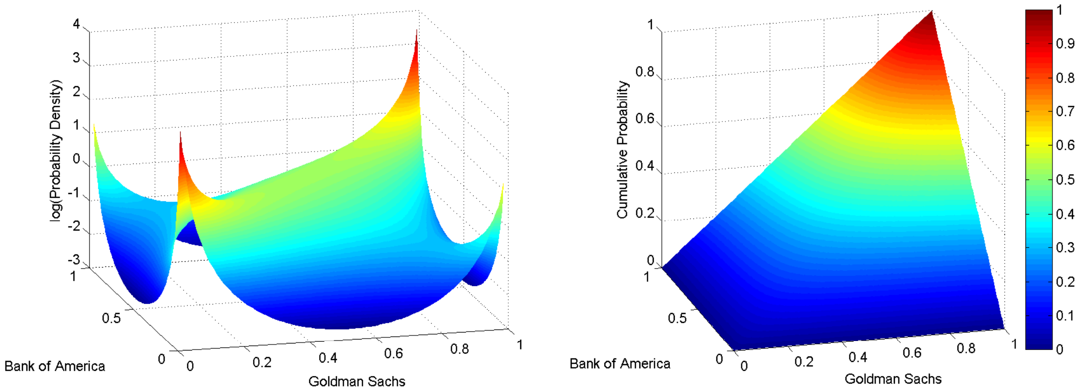


Figure 3.20: Probability and cumulative distribution functions, for Goldman Sachs vs Bank of America, before the crisis, T copula.

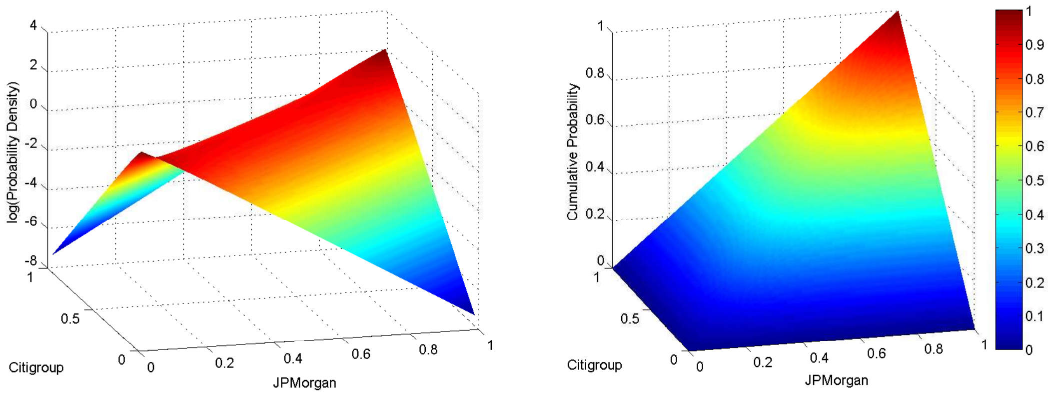


Figure 3.21: Probability and cumulative distribution functions, for JPMorgan vs Citigroup, during the crisis, Frank copula.

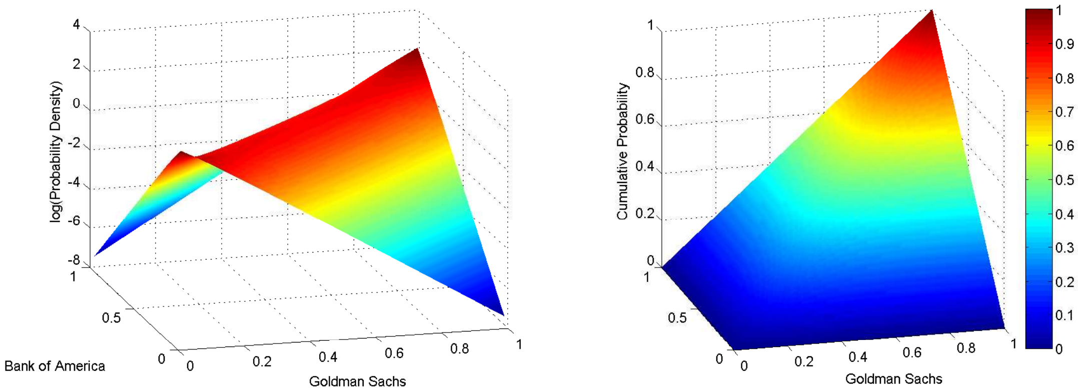


Figure 3.22: Probability and cumulative distribution functions, for Goldman Sachs vs Bank of America, during the crisis, Frank copula.

Due to the over-reactiveness nature of the CDS data, in particular during the crisis, we also carried out a sensitivity analysis for all the results, since the marginal estimations until the joint distributions using copula functions. The CDS spreads variations were truncated at three times the standard deviation of the corresponding series, for each bank. Then, the ARMA-GARCH modelling was repeated and we did not obtain substantial differences, neither in the significant parameters' estimates nor in the residuals. The copula procedures were also repeated and the same copula (Frank) would be chosen for the crisis period according to AIC criterion, for all the pairs of institutions. Furthermore, the copula parameters and the dependency measures present negligible differences, maintaining the rank of dependency measures for the generality of the pairs, compared with the results depicted in the present work using the original CDS spreads variations. Therefore, we conclude that our results are robust to the existence of outliers in the CDS data, resultant of the announcements, news and bailouts in the crisis period.

Interpretation of the results within the crisis framework

The dependency measures before and during the crisis can be interpreted according to the selected copulas. We notice a substantial increase in the global dependency measures (see tables 3.34 and 3.35) during the crisis, indicating an increase of the contagion risk, as it was expected under the stress scenario of a crisis period. The least correlated pairs are Goldman Sachs vs Bank of America and Goldman Sachs vs Citigroup, before and during the crisis, respectively. The corresponding Kendall's tau goes from 0.398 and 0.464 in the pre-crisis period to 0.663 and 0.645 during the crisis.[14] It was expected that the least associated pairs were composed by Goldman Sachs, due to the fact that the institution left their statute of investment bank only in the peak of the crisis. The most dependent pair, in both periods, was JP Morgan vs Bank of America, the top two US banks in 2008, according to BANKERSAlmanac ranking. Their dependency measures increased substantially in the crisis period (Kendall's tau goes from 0.574 to 0.719), for which may have contributed their acquisitions of two investment banks in 2008, Bear Stearns and Merrill Lynch, respectively, turning their business profiles more similar. Recalling the correlations of the standardized residuals (tables 3.23 and 3.24), which considered the pairs Bank of America vs Citigroup and JPMorgan vs Citigroup as the most dependent pairs before and during the crisis, respectively, instead of JP Morgan vs Bank of America, according to Kendall's tau and Spearman's rho. Therefore, the different rank of dependencies might be due to the fact that correlations are more dependent on extreme values than the rank correlations measures, affecting the results.

Table 3.34: Dependency measures according to the T copula, before the crisis.

T	Kendall's tau	Spearman's rho	λ_u	λ_l
JP Morgan vs Goldman Sachs	0.570	0.765	0.287	0.287
JP Morgan vs Bank America	0.574	0.770	0.516	0.516
JP Morgan vs Citigroup	0.561	0.757	0.599	0.599
Goldman Sachs vs Bank America	0.398	0.568	0.511	0.511
Goldman Sachs vs Citigroup	0.464	0.648	0.515	0.515
Bank America vs Citigroup	0.496	0.686	0.597	0.597

Table 3.35: Dependency measures according to the Frank copula, during the crisis.

Frank	Kendall's tau	Spearman's rho	λ_u	λ_l
JP Morgan vs Goldman Sachs	0.663	0.858	0.000	0.000
JP Morgan vs Bank America	0.719	0.901	0.000	0.000
JP Morgan vs Citigroup	0.660	0.856	0.000	0.000
Goldman Sachs vs Bank America	0.663	0.858	0.000	0.000
Goldman Sachs vs Citigroup	0.645	0.842	0.000	0.000
Bank America vs Citigroup	0.687	0.877	0.000	0.000

[14] The Spearman's rho is also ordered in the same way.

Moreover, in the context of the market panic and bailouts occurred in the crisis period, we expected that the selected copula provides tail dependency, in particular, upper tail dependency. Intuitively, this means that more probability mass should be assigned to events in the right tail (related to increases in CDS spreads variation and simultaneous defaults) during the crisis. Although the selected copula according to the AIC criterion, the Frank copula, has neither lower nor upper tail dependency, we can analyse the tail dependency by considering other estimated specifications. In fact, the parameter estimates of the Gumbel copula are significant so, we can interpret the respective upper tail dependency coefficients in both periods. T-copula also has tail dependency, both upper and lower. Nevertheless, as we saw before, the degrees of freedom are not statistically significant in some cases. Thus, for the sake of consistency, we study the Gumbel copula for all pairs (table 3.36). All the upper tail coefficients increased in the crisis period, for each pair, as expected. The most dependent pair according to rank correlation measures (JPMorgan vs Bank of America, the non-directly bailed out institutions of our sample) exhibited also the highest upper tail dependency coefficient during the crisis, 0.77, evidencing a significant probability of simultaneous large increases on CDS spreads variations (in the right tail) of these institutions. This confirms the panic and the effect of bad results/performances in financial markets. Regarding to the first period, the most dependent pair in the right tail was Bank of America vs Citigroup, that changed to the second position in the crisis period. The minimum values of upper tail dependency coefficients, in both periods, belong to the pairs composed by Goldman Sachs that, once again, might be due to the different business profile until September 2008.

Table 3.36: Upper tail dependency coefficients according to the estimated Gumbel copula, before and during the crisis, respectively.

λ_u	Pre-crisis	Crisis period
JP Morgan vs Goldman Sachs	0.628	0.704
JP Morgan vs Bank America	0.665	0.773
JP Morgan vs Citigroup	0.679	0.711
Goldman Sachs vs Bank America	0.604	0.693
Goldman Sachs vs Citigroup	0.598	0.685
Bank America vs Citigroup	0.685	0.726

Chapter 4

Summary and conclusions

The aim of this thesis is to analyse the default dependency structure of four US financial institutions, JP Morgan Chase & Co., Goldman Sachs Inc, Bank of America Corp and Citigroup Inc., using conditional copulas to evaluate the default dependency in two periods: before the crisis, since 2006 until the summer of 2007; and thereafter until March 2009, the crisis period. The dependency was assessed for the six pairs of the aforementioned financial institutions in the both periods.

Our empirical study relies on CDS data, which is assumed as a proxy for default closeness of the institutions. This is a more direct indicator than the widely used equity returns for default dependency purposes. The CDS levels of the considered institutions, as well as their correlations, increased substantially in the crisis period. According to ADF test, as these processes are first order integrated, they cannot be directly used to copula fitting. The first differentiation of CDS spreads, the CDS spreads variations, were proper candidates as input, as they proved to be stationary series. The hypothesis of normality was rejected, according to QQ-plots and Jarque-Bera Statistic, and there was evidence of conditional heteroscedasticity pointed out by ARCH test, as it is usual in financial series. The correlograms suggested an AR(1)-GARCH(1,1) model for the generality of the banks and for both periods. The assumption of a t-distribution for innovations overperform the Gaussian assumption, according to the AIC criterion. In fact, the estimates indicate an IGARCH model, with absence of autocorrelations and conditional heteroscedasticity on residuals for the generality of the cases, pointed out by Ljung-Box and ARCH tests, respectively. Only the contemporaneous correlations remains for almost all the considered pairs, presenting higher values during the crisis than in the pre-crisis period. Therefore, given that the marginal behaviour of CDS spreads daily changes are well specified, the iid residuals are recovered and transformed to uniform variables through the t-cumulative distribution function. After this procedure, it is possible to implement the copula framework, conditioned on the previous results. Gaussian, T, Frank, Clayton and Gumbel copulas were fitted through IFM method and the selection was based on AIC criterion. Analysing the AIC values and the confidence intervals to their parameters, the most suitable copula functions are the T and the Frank for the generality of the cases, before and during the crisis, respectively. The selection of two different copulas, in the two sub-periods, and the increase in the dependency measures point out for a structural break on the default dependency patterns, in the summer of 2007, as it was expected. The most dependent pair in both periods was JP Morgan vs Bank of America, the top two banks in 2008, whose their acquisitions of two investment banks in the crisis turned their business profile more similar. The least dependent pairs, according to the rank correlations measures, were composed by Goldman Sachs, the ex-investment bank. In line with the functional forms of the selected copulas, there was no tail dependency in the crisis period, which was not expected. However, we can analyse other copula specifications that do not have the optimal AIC but their parameters are significant for all cases, such as the Gumbel copula, which has implied a positive upper tail dependency measure. The upper tail dependency coefficient increased substantially in the crisis period, confirming the market panic and the lack of discrimination on turbulent moments. In the crisis period, the most dependent pair in the case of simultaneous defaults is JP Morgan vs Bank of America, which is also the most global dependent pair, while the least upper tail coefficient belongs to Goldman Sachs vs Citigroup, in both periods.

An alternative modelling of default dependency may be performed using the dynamic copula functions (as described in appendix A). The conclusions are very different depending on the assumptions made for the time-varying parameter. Hence, as future work, a set of time-varying specifications should be tested in order to provide reliable results based on dynamic copulas. Another line of research is to consider copula mixture models or time-varying copulas with switching regimes allowance to take into account the structural breaks, as suggested by Rodriguez (2007). All the aforementioned specifications should also be applied to other institutions, for example, sovereigns or banks of different countries, to evaluate measures of dependency/spillovers between different markets. Other interesting applications could also be done within this framework, for instance, value at risk of portfolios and credit risk products pricing, as in Palaro and Hotta (2006) and Schönbucher and Schubert (2001), respectively.

To conclude, we can not fail to mention some lessons that can be taken from the crisis, beyond all the papers and mathematical results. These are fundamentally based on experience and should be used as a complementary tool, to prevent other problematic times:

- ⋆ *Do not invest in instruments you do not understand.* Translate the quantitative approaches to pricing structured products into simple and intuitive explanations.

- ⋆ *Understand your risk exposure.* Identify exposures and concentrations, make a integrated assessment of risk and its sources, perform stress testing and worst case scenario analysis.

- ⋆ *Use a comprehensive set of indicators.* Complement rating information with more reactive indicators based on market information. Ratings take time to adjust due to their *through the cycle* perspective.

- ⋆ *Be aware of indirect exposures.* Pay attention to the counterparty risk.

- ⋆ *No return without risk.* Evaluate the extra risk that comes from the eventual extra return and decide if you want to take it.

- ⋆ *Too big to fail is false!* With this crisis, we understood that the complex financial system is fragile. If a bank as Lehman Brothers can fail, who is safe?

Bibliography

BIS (2009), 79th annual report. Bank for International Settlements.

Bluhm, C. (2003), CDO modeling: Techniques, examples and applications. Hypo Vereinsbank, Structured Finance Analysis.

Cherubini, U., Luciano, E. and Vecchiato, W. (2004), *Copula Methods in Finance*, John Wiley & Sons, England.

Dias, A. and Embrechts, P. (2003), Dynamic copula models for multivariate high-frequency data in finance. Working Paper, ETH Zurich: Department of Mathematics.

Donnelly, C. and Embrechts, P. (2010), The devil is in the tails: Actuarial mathematics and the subprime mortgage crisis. RiskLab, ETH Zurich.

Duffie, D. and Singleton, K. J. (2003), *Credit Risk: Pricing, Measurement and Management*, Princeton Series in Finance.

Elizalde, A. (2005), Credit risk models I: Default correlation in intensity models. Available at www.abelelizalde.com.

Embrechts, P. A., Hoing, A. and Juri, A. (2003), 'Using copulae to bound the value-at-risk for functions of dependent risks', *Finance and Stochastics* **7**(2), 145–167.

Embrechts, P., Lindskog, F. and McNeil, A. (2001), Modelling dependence with copulas and applications to risk management. Working Paper CH-8092 ETH Zurich: Department of Mathematics.

Embrechts, P., McNeil, A. J. and Straumann, D. (1999), 'Correlation: Pitfalls and alternatives', *Risk Magazine* **12**(May), 69–71.

Galiani, S. S. (2003), Copula functions and their application in pricing and risk managing multiname credit derivative products. Report submitted in partial fulfillment of the requirements for the degree of MSc in Financial Mathematics in the University of London.

Genest, C., Gendron, M. and Bourdeau-Brien, M. (2009), 'The advent of copulas in finance', *The European Journal of Finance* **15**(7–8), 609–618.

Hitier, S. and Huber, E. (2009), CDO pricing: Copula implied by risk neutral dynamics. Available at www.defaultrisk.com.

Hull, J. and White, A. (2006), 'Valuing credit derivatives using an implied copula approach', *Journal of Derivatives* **14**(2), 8–28.

Hull, J. and White, A. (2007), 'Forward and european options on CDO tranches', *Journal of Credit Risk* **3**(2), 63–73.

Hull, J. and White, A. (2008), 'Dynamic models of portfolio credit risk: A simplified approach', *Journal of Derivatives* **15**(4), 9–28.

Joe, H. and Xu, J. J. (1996), The estimation method of inference functions for margins for multivariate models. Dept. of Statistics, University of British Columbia.

Lando, D. (2004), *Credit Risk Modeling: Theory and Applications*, Princeton University Press.

Li, D. X. (2000), 'On default correlation: A copula function approach', *Journal of Fixed Income* **9**(4), 43–54.

Mashal, R. and Zeevi, A. (2002), Beyond correlation: Extreme co-movements between financial assets. Working Paper, Columbia University.

McNeil, A. J., Frey, R. and Embrechts, P. (2005), *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton University Press, New Jersey.

Meneguzzo, D. and Vecchiato, W. (2002), Copula sensivity in collateralized debt obligations and basket default swap pricing and risk monitoring. Risk Management Department, Intesa Bank, Milan, Working Paper.

Nelsen, R. B. (1999), *An Introduction to Copulas: Lecture Notes in Statistics*, Vol. 139, Springer-Verlag, New York.

Nelson, D. B. (1990), 'Stationarity and persistence in the GARCH(1,1) model', *Econometric Theory* **6**, 318–334.

Palaro, H. P. and Hotta, L. K. (2006), 'Using conditional copula to estimate value at risk', *Journal of Data Science* **4**, 93–115.

Patton, A. J. (2004), 'On the out-of-sample importance of skewness and asymmetric dependence for asset allocation', *Journal of Financial Econometrics* **2**(1), 130–168.

Rodriguez, J. C. (2007), 'Measuring financial contagion: A copula approach', *Journal of Empirical Finance* **14**(3), 401–423.

Romano, C. (2002), Calibrating and simulating copula functions: An application to the italian stock market. Working Paper 12/2002 of CIDEM.

Schönbucher, P. J. (2003), *Credit Derivatives Pricing Models: Model, Pricing and Implementations*, John Wiley & Sons.

Schönbucher, P. J. and Schubert, D. (2001), Copula-dependent default risk in intensity models. Working Paper, Department of Statistics, Bonn University.

Appendix A

Dynamic copulas - An illustration

The dynamic copulas take into account the time-variability of some parameter, such as the linear correlation, Kendall's τ or the copula parameter. Although, at a first sight, one could think that the dynamic copulas would always overperform the static ones, this conclusion depends on the specification chosen for the dynamic copula. In this appendix, we present a dynamic Gumbel copula, comparing the results with those obtained for the static Gumbel copula during the crisis (see subsection 3.2.2).

There are many ways of capturing possible time variation in the conditional copula formulation. We will assume that the functional form of the copula remains fixed over the sample while the parameters vary according to some evolution equation. An alternative to this approach is to allow also for time variation in the functional form using a regime switching copula model, as presented in Rodriguez (2007).

Consider the Gumbel copula and let u, v uniform variables,

$$\mathbf{C}(\theta_t(u,v)) = \exp\{-[(-\ln u)^{\theta_t} + (-\ln v)^{\theta_t}]^{1/\theta_t}\}.$$

Assume that the parameter of the Gumbel copula varies according to the following equation:[1]

[1]This evolution is suggested by Andrew Patton that created the MATLAB codes for this dynamic Gumbel copula, available on http://econ.duke.edu/∼aep172/. It is clear that an infinity of specifications can be assumed for the parameter and the results completely depend on that. The aim of this appendix is only to provide an illustration of this dynamic tool.

$$\theta_t = 1 + \left(\omega + \beta\theta_{t-1} + \alpha \frac{1}{10} \sum_{j=1}^{10} |u_{t-j} - v_{t-j}| \right)^2 \tag{A.1}$$

The conclusions presented below are based on the AIC criterion, for the same six pairs of institutions (JPMorgan vs Goldman Sachs, JPMorgan vs Bank of America, JPMorgan vs Citigroup, Goldman Sachs vs Bank of America, Goldman Sachs vs Citigroup and Bank of America vs Citigroup). Considering the whole sample, *i.e.*, the two periods together, we recalculated the AR(1)-GARCH(1,1) estimates. The dynamic copula presents smaller AIC values than the static one for the generality of the cases, with the exceptions being the pairs JP Morgan vs Goldman Sachs and JP Morgan vs Citigroup.[2] However, as there is empirical evidence of a structural break, given that the selected copulas before and during the crisis have different specifications, the use of the same copula for the whole period is a limitation, as it does not consider the change of the dependency structure. Therefore, estimations are carried out for the more recent sub-period considered in this study. According to the AIC criterion, results point out to a better performance of the time-varying Gumbel regarding to the static Gumbel. The exceptions were the same pairs obtained before, in the whole sample estimation. The figures A.1 to A.6 present, on the left-hand side, the parameters of the Gumbel time varying-copula and the one of the static copula; on the right-hand side, the corresponding Kendall's tau, for all the pairs considered. As we expect, the static parameters (and the associated Kendall's tau) lie in the middle of the respective time-varying estimates along the period.

Summing up, either in the crisis period or in the whole period, the dynamic copula with the suggested specification does not present smaller AIC values than the corresponding static copula for all pairs. For two of them, the static Gumbel is better. However, more specifications have to be tested to take reliable conclusions based on time-varying copulas. This is a line of research to be pursued.

[2]The likelihoods and estimates for parameters considering the whole sample are not presented here in order to save space, given that this approach is not the aim of the thesis.

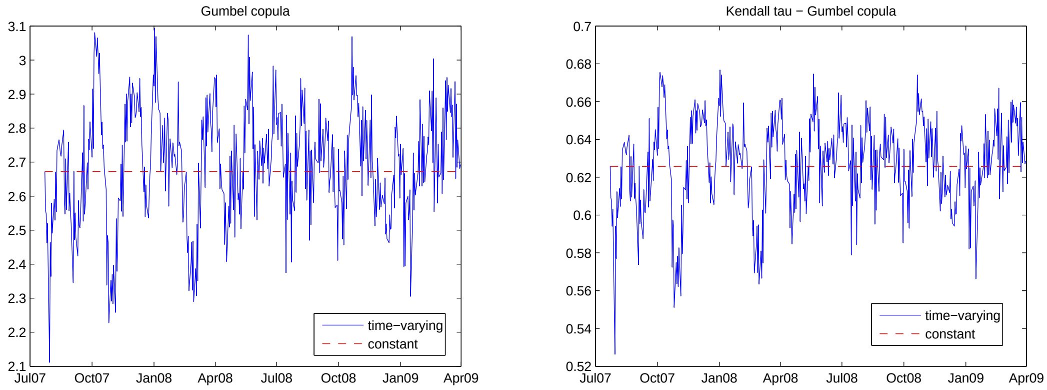


Figure A.1: Time-varying parameter and Kendall's tau, for JP Morgan vs Goldman Sachs, during the crisis, Gumbel copula.

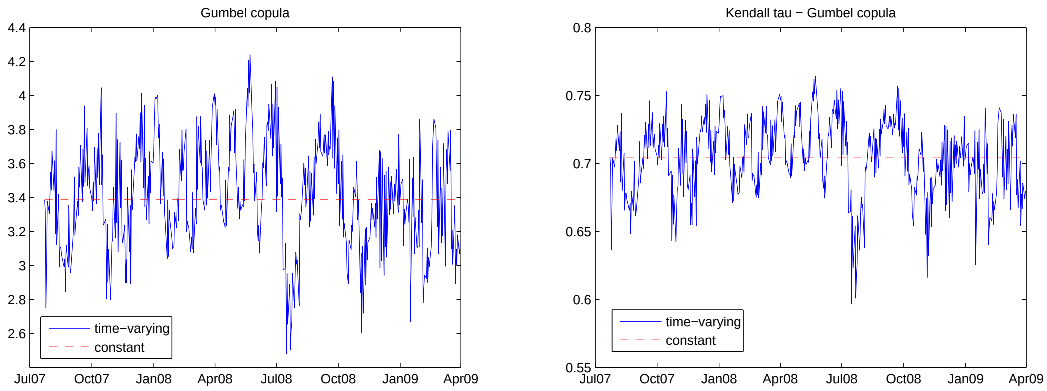


Figure A.2: Time-varying parameter and Kendall's tau, for JP Morgan vs Bank of America, during the crisis, Gumbel copula.

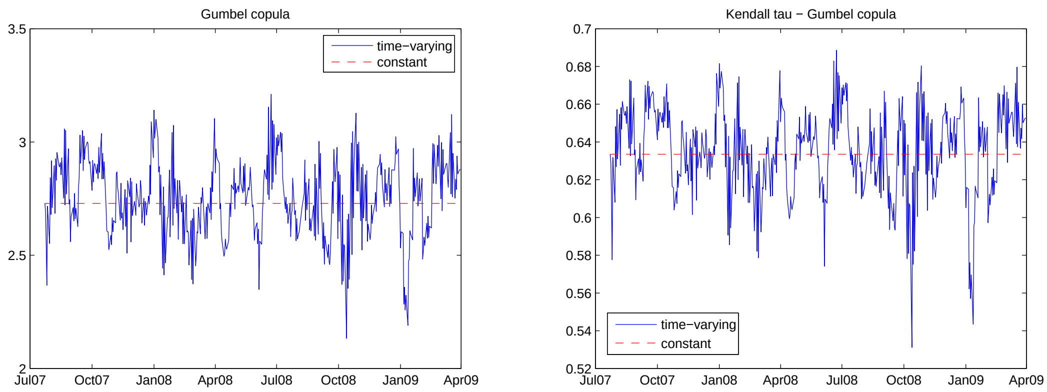


Figure A.3: Time-varying parameter and Kendall's tau, for JP Morgan vs Citigroup, during the crisis, Gumbel copula.

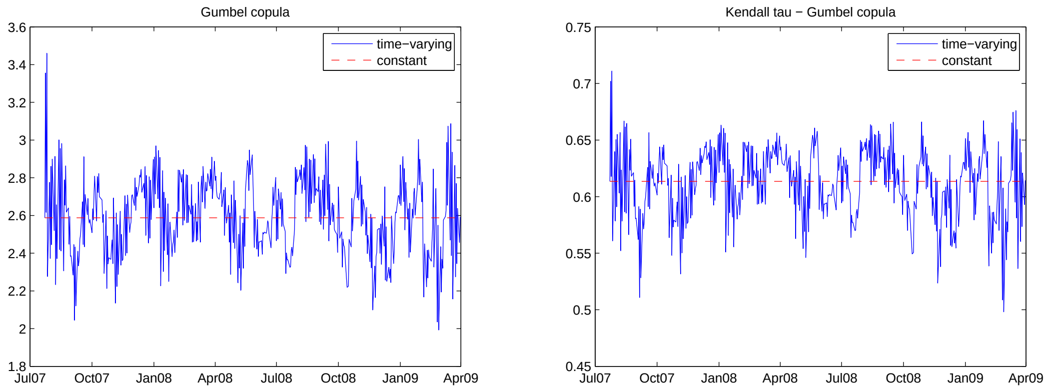


Figure A.4: Time-varying parameter and Kendall's tau, for Goldman Sachs vs Bank of America, during the crisis, Gumbel copula.

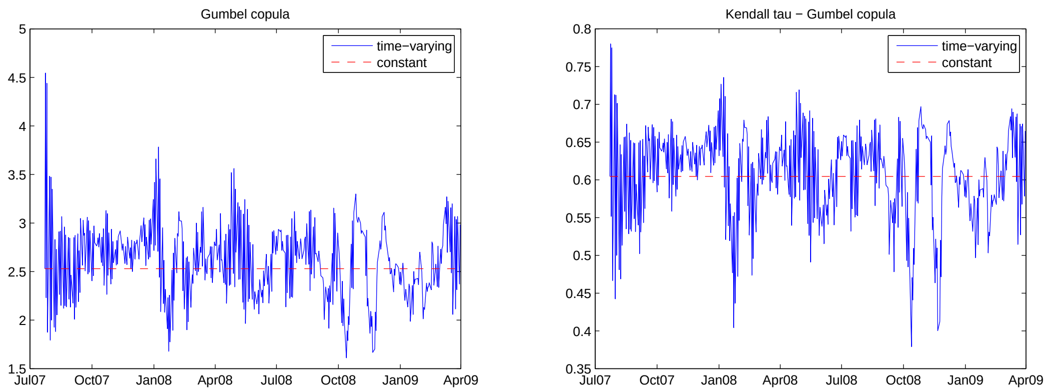


Figure A.5: Time-varying parameter and Kendall's tau, for Goldman Sachs vs Citigroup, during the crisis, Gumbel copula.

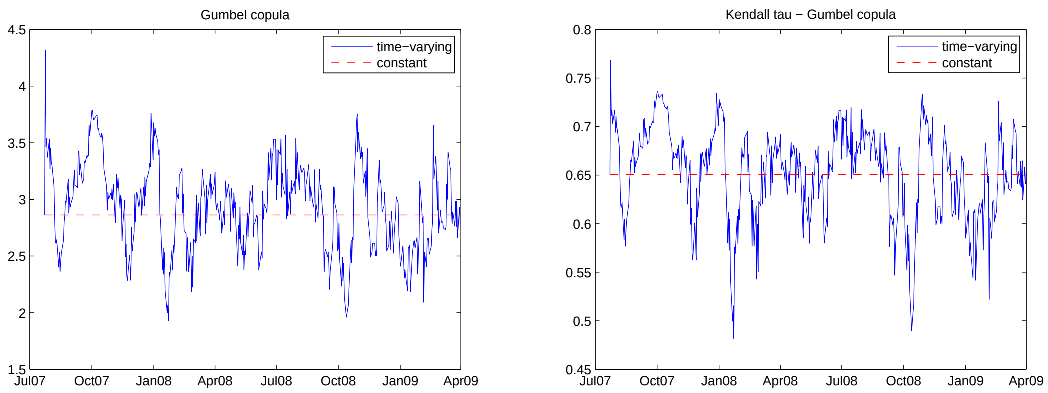


Figure A.6: Time-varying parameter and Kendall's tau, for Bank of America vs Citigroup, during the crisis, Gumbel copula.